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A COURSE ON THE SOLUTION OF SPHERICAL TRIANGLES

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FOR THE

MATHEMATICAL LABORATORY

BY

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PREFACE

THE aim of the following pages is to present, in a form suitable for the use of students in the Mathematical Laboratory, an account of the various methods of solution of the spherical triangle, numerical and graphical.

The subject is of importance in view of its applications. It has also considerable value from the educational point of view; it develops the power of dealing with "geometry of situation," and at the same time provides an excellent training in computation and furnishes instructive comparisons of the accuracy of different methods of solution.

I wish to express my thanks to Professor Whittaker, who invited me to undertake the work and has helped me with many valuable suggestions; to Mr A. W. Young, M.A., B.Sc., for reading the manuscript and proofs; and to Mr E. L. Ince, M.A., B.Sc., for drawing the greater part of the diagrams.

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CHAPTER I

THE USE OF LOGARITHMIC TABLES

§ 1. **Introductory.**—Most of the calculation throughout this tract involves the use of logarithms, and it is important that the student should be so expert in their use that he can carry out the calculations indicated with a minimum of trouble.

The number of decimal places required in the logarithm depends on the accuracy aimed at; as a rough guide, it may be said that four-place tables are sufficient when solving to an accuracy of minutes, five places when solving to an accuracy of five seconds, and that six-place tables enable us to work to half-seconds.

As regards the accuracy needed in various applications of trigonometry, we may roughly say that in topography and navigation an accuracy of a half-minute is sufficient, and that in geodesy and astronomy the limits of accuracy are a second and a tenth of a second respectively.

Since the angles concerned occur in all four quadrants, the sine, cosine, or tangent is very frequently negative, and therefore has to be multiplied by -1 if its logarithm is to have a real value. This is indicated by writing the letter n after the logarithm of the positive number. The n's of two negative numbers which are multiplied together, or divided, evidently cancel each other.

To find the logarithm of the trigonometrical function (sine, cosine, tangent, etc.) of any angle in the tables (which, of course, only extend to 90°) we subtract (or add if negative) multiples of 90°, the function being unchanged if the multiple is even, but becoming the co-function if the multiple is odd.

§ 2. **Calculations with Seven-place Tables.**—For the most part seven-place tables will be used in the following pages, the

shorter calculation where fewer figures are required being evident. It will be assumed that the student has access to tables giving the logarithmic sines, cosines, and tangents for every ten seconds.

In a modern book of seven-place tables we meet with entries like $\log 2 = 0{\cdot}301030\underline{0}$. The stroke under the last 0 means that the exact value lies somewhere between 0·30102995 and 0·30103000, so that the most probable value is 0·3010299,75. If we wish to obtain the seventh digit of our final result as accurately as is possible with the use of seven-place tables, we can use this most probable value until the completion of the operation, afterwards returning to seven places. Similarly, where we find $\log 5 = 0{\cdot}6989700$, we can use 0·6989700,25 as being the most probable value, the absence of the stroke showing that the exact value exceeds the given one. The following examples will make the procedure clear.* As a general rule, however, if it is important to have the seventh digit accurate in the final result, we should use tables to more than seven places.

Example 1.—*To find log cot* 124° 17′ 10″·24.

This angle being in the second quadrant, the cotangent is negative.

$$\log\cot 124^\circ\, 17'\, 10''{\cdot}24 = -\log\tan 34^\circ\, 17'\, 10''{\cdot}24$$

$$\begin{array}{lrr}
\text{L}\tan & 34^\circ\, 17'\, 10'' & 9{\cdot}8336\;561{,}25 \\
 & {\cdot}2 & 9{,}04 \\
 & {\cdot}04 & 1{,}808 \\
\hline
\log\tan & 34^\circ\, 17'\, 10''{\cdot}24 = & 9{\cdot}8336\;572{,}1 \\
\therefore\quad \log\cot & 124^\circ\, 17'\, 10''{\cdot}24 = & 9{\cdot}8336\;572{,}1n.
\end{array}$$

One digit after the comma is retained not as being correct, but as the most probable.

Example 2.—*Find log sin* 340° 10′ 11″·81.

Enter for log cos 70° 10′ 11″·81.

Since the cosines and cotangents decrease as the argument increases, we take from the table the entry corresponding to the argument next *above* and subtract fractions of 10″ so as to make the differences *additive*. The scheme on the left side is first formed by use of complementary numbers; the logarithms are then entered on the right.

$$\begin{array}{lrr}
 & 70^\circ\, 10'\, 20'' & 9{\cdot}5304482{,}25 \\
 & -8 & 467{,}2 \\
 & -\ {\cdot}1 & 5{,}84 \\
 & -\ {\cdot}09 & 5{,}256 \\
\hline
\log\cos & 70^\circ\, 10'\, 11''{\cdot}81 = & 9{\cdot}5304960{,}6 \\
\log\sin & 340^\circ\, 10'\, 11''{\cdot}81 = & 9{\cdot}5304960{,}6n
\end{array}$$

* The data in the following examples have been taken from the seven-figure logarithmic tables of Schrön, which are used in the Mathematical Laboratory of Edinburgh University.

Example 3.—*Find θ from*

$\log \sin \theta = 9{\cdot}5281423{,}0n$	
$9{\cdot}5281053{,}25$	$-19^\circ\ 43'\ 0''$
$369{,}75$	
$352{,}8$	6
$16{,}95$	
$11{,}76$	$\cdot 29$
$5{,}190$	$-19^\circ\ 43'\ 6''\ \cdot 29 = \theta$

i.e. $\theta = 340^\circ\ 16'\ 53''{\cdot}71$ or $199^\circ\ 43'\ 6''{\cdot}29$

Example 4.—*Find θ from*

$\log \cos \theta = 9{\cdot}7137204{,}0$	
$259{,}75$	$58^\circ\ 51'\ 0''$
$-55{,}75$	
$-34{,}8$	$1''$
$-20{,}95$	
$-20{,}88$	$0''{\cdot}6$
$-\ 0{,}07$	
	$\theta = 58^\circ\ 51'\ 1''{\cdot}60$
	or $301^\circ\ 8'\ 58''{\cdot}40$

The procedure is somewhat more difficult when the tabular differences become so great that proportional part calculation is no longer accurate to the seventh place, as happens with the logarithmic sines and tangents of very small angles and the logarithmic cosines of angles near a right angle. In such cases additional tables are given, the S tables for the sines and the T tables for the tangents. We first convert the angle to seconds and find the logarithm of the resulting number, and to this is added the value given in the S or T table for the given angle, the sum being the required logarithm. The method of entering the S or T tables for the required value is fully explained in the introduction to Schrön.

§ 3. **Choice of Function.**—When we have the choice of determining an angle from its sine, its cosine, or its tangent, we prefer the tangent, as its tabular differences are everywhere greater than either of the other two. When the choice lies between sine and cosine, the same argument leads us to choose whichever is the smaller. Conversely, when we require to find the logarithm of one of these functions, the tangent is the most difficult to get accurately to, say, seven places, and of the sine or cosine, that one is the more difficult which is the smaller. For example, in certain calculations (§ 46) it is required to solve equations of the form

$$x \sin y = a$$
$$x \cos y = b.$$

Here, therefore, we first find y from $\tan y = a/b$, and then find x by using whichever is the greater, $\sin y$ or $\cos y$.

Example 5.—*Find x and y from*

$$x \sin y = 2{\cdot}417132$$
$$x \cos y = -8{\cdot}141601.$$

We have

$$\log(x \sin y) = 0{\cdot}3833\ 003{,}9$$
$$\log(x \cos y) = 0{\cdot}9107\ 098{,}3n$$

$$\mathrm{L}\tan y = 9{\cdot}4725\ 905{,}6n \qquad \therefore\quad y = -16^\circ\ 32'\ 7''{\cdot}61$$
$$\mathrm{L}\cos y = 9{\cdot}9816\ 572{,}7$$
$$\log x = 0{\cdot}9290\ 525{,}6n \qquad x = -0{\cdot}8492\ 833$$

§ **4. Logarithms of Addition and Subtraction.***—One of the commonest operations in computation is the determination of $\log(p+q)$ when $\log p$ and $\log q$ are known. This may, of course, be done by finding the antilogarithms of $\log p$ and $\log q$, and hence obtaining the logarithm of their sum; it is, however, often convenient to make use of a table of addition-logarithms † for this purpose. This is simply a table of the function $\log(1+1/x)$ corresponding to the argument $\log x$; if we denote p/q by x (where $p>q$), we can calculate $\log x$ from the equation

$$\log x = \log p - \log q,$$

and then, obtaining the value of $\log(1+1/x)$ from the table of addition-logarithms, we have $\log(p+q)$ from the equation

$$\log(p+q) = \log p + \log(1+1/x).$$

The differences in the table are negative, so that we proceed as in using the cosine or cotangent tables (*cf.* Example 2 above).

Example 6.—Given $\log x = 0{\cdot}4162\ 147$, find $\log(1+1/x)$.

From the table we have the corresponding tabular values

4163	1409 610
8	221,6
5	13,85
3	831
4162 147	1409 846,3

The computation of $\log(p+q)$ should be arranged as in the following example :—

* Called also Gaussian Logarithms, Gauss having prepared (1812) the first tables (five places). The invention is, however, due to Leonelli (1803).

† The best six-place table is that of B. Cohn, *Tafeln d. Add. u. Sub. Log.*, Engelmann, Leipzig (1909). For seven places we have that of J. Zech, *Tafeln d. Add. u. Sub. Log.*, Engelmann, Leipzig (1849).

Example 7.—Given $\log p = 0\cdot6685\ 166$, $\log q = 0\cdot2523\ 019$, to find $\log(p+q)$.

$$\begin{array}{rll}
\log p = & 0\cdot6685\ 166 & \text{(given)} \\
\text{addit. } \log = & 0\cdot1409\ 846 & \text{(the tabular value corresponding to line 4)} \\
\log q = & 0\cdot2523\ 019 & \text{(given)} \\
\log x = & \underline{0\cdot4162\ 147} & (\log p - \log q) \\
\log(p+q) = & 0\cdot8095\ 012 & \text{(adding lines 1 and 2).}
\end{array}$$

Similarly, in order to find $\log(p-q)$ when $\log p$ and $\log q$ are known, we write $x = p/q$, and then have

$$\log(p-q) = \log p - \log \frac{1}{1-1/x}.$$

The values of $\log \dfrac{1}{1-1/x}$ corresponding to the argument $\log x$ are given in a *table of Subtraction-logarithms*.

Example 8.—Given $\log x = 0\cdot4162\ 147$, to find $\log(1-1/x)$.

By the Subtraction table we have

$$\begin{array}{rr}
4162\ 147 & \\
\underline{297} & 2100\ 7 \\
-150 & \\
\underline{-144{,}9} & 9 \\
5{,}1 & \\
\underline{4{,}8} & \underline{3} \\
 & 2100\ 793
\end{array}$$

Example 9.—With the same values of $\log p$ and $\log q$ as in Example 7, to find $\log(p-q)$.

$$\begin{array}{rll}
\log p = & 0\cdot6685\ 166 & \text{(given)} \\
\text{sub. } \log = & 0\cdot2100\ 793 & \text{(the tabular value for line 4)} \\
\log q = & 0\cdot2523\ 019 & \text{(given)} \\
\log x = & \underline{0\cdot4162\ 147} & (\log p - \log q) \\
\log(p-q) = & 0\cdot4584\ 373 & \text{(subtracting line 2 from line 1).}
\end{array}$$

Example 10.—Find a from $\cos a = \cos b \cos c + \sin b \sin c \cos A$, where $b = 49°\ 24'\ 10''\cdot23$, $c = 38°\ 46'\ 10''\cdot35$; $A = 110°\ 51'\ 14''\cdot84$.

$$\begin{array}{rl@{\qquad}rl}
L\cos b = & 9\cdot8134\ 052{,}1 & L\sin b = & 9\cdot8804\ 154{,}6 \\
L\cos c = & \underline{9\cdot8919\ 113{,}6} & L\sin c = & 9\cdot7967\ 057{,}4 \\
\log p = & 9\cdot7053\ 165{,}7 & L\cos A = & \underline{9\cdot5514\ 376{,}1n} \\
 & \underline{9\cdot2285\ 588{,}1n} & \log q = & 9\cdot2285\ 588{,}1n \\
\log x = & \cdot4767\ 577{,}6n & &
\end{array}$$

$$\begin{array}{rr}
\underline{640} & \cdot1762\ 7 \\
-624 & \\
\underline{-600} & 3 \\
-\ 240 & \underline{1} \\
 & \cdot1762\ 731 = \text{subtr. log.} \\
\log p = & \underline{9\cdot7053\ 166} \\
L\cos a = & 9\cdot5290\ 435
\end{array}$$

$$a = 70°\ 14'\ 20''\cdot12.$$

Examples:—

(1) Verify by means of the tables the following:—

$$\begin{aligned} &\mathrm{L}\sin\ 16^\circ\, 45'\, 38''\!\cdot\!21 = 9\!\cdot\!4599\ 555{,}6 \\ &\mathrm{L}\tan\ 275^\circ\, 14'\, 28''\!\cdot\!42 = 1\!\cdot\!0374\ 758{,}5n \\ &\mathrm{L}\cos\ 92^\circ\, 10'\, 11''\!\cdot\!24 = 8\!\cdot\!5781\ 910{,}6n. \end{aligned}$$

(2) Taking the above logarithms as given, find the corresponding angles.

(3) Verify by addition-logarithms the following identities—

$$2+3=5\,;\ 7-6=1\,;\ -5+4=-1\,;\ -1-1=-2.$$

CHAPTER II

SPHERICAL TRIANGLES

§ 5. **Circles on a Sphere and their Terminology.**—When a plane passes through the centre of a sphere the intersection of the plane and sphere is called a *Great Circle.* If the plane does not pass through the centre it cuts the sphere, if at all, in a *Small Circle.*

Consider the intersections of a series of parallel planes with the sphere. Symmetry shows that the centres of all the circles of intersection (fig. 1) will lie on a *straight line NS*—the *axis* —which is perpendicular to all the planes. Two of the series of parallel planes will intersect the sphere in circles of zero radius—in other words, will touch the sphere. Their points of contact will be the ends of the axis and are called the *poles* of the circles. A pole is clearly equidistant from all points on the circle of which it is a pole, and thus corresponds in a certain sense to the centre of a circle in plane geometry. The particular circle of the series which is a great circle is called the *equator.* It is at a quadrant's distance from each of its poles.

Each circle has thus *two* poles. It is convenient, however, to define a *unique* pole by the following convention: we give the circle a "sense," describing motion in one direction round the circle as "positive" and motion in the opposite direction as "negative." *The* pole of the circle is then described as that one which would be towards the left-hand side of a man walking on the sphere in the positive direction round the circle. Another mode of describing *the* pole is to say that an observer situated at the pole would regard the positive direction in the circle as anti-clockwise.

If we take now a series of planes through the axis, their intersections with the sphere are a series of great circles which all pass through the poles of the first series of circles and which

intersect every circle of the first series at right angles. These are called *Meridian Circles.*

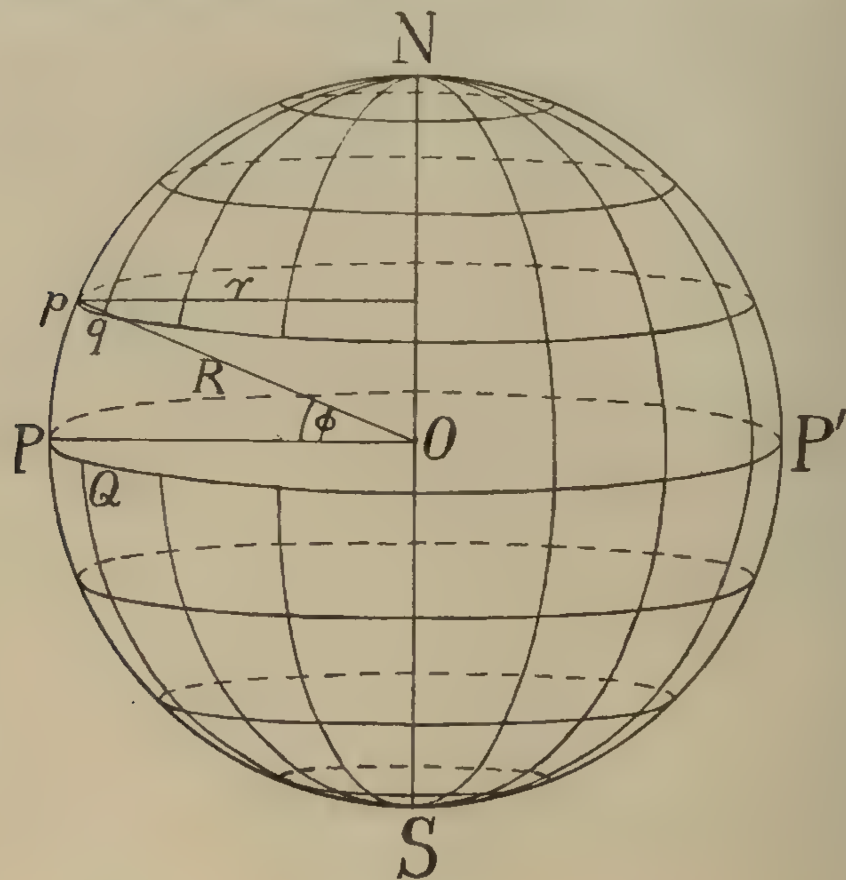

FIG. 1.

The region on the sphere which is enclosed between any two semi-great circles—*e.g.* two meridians—is called a *Lune.*

§ 6. **Coordinates on a Sphere.**—It will be seen that the two series of circles with which we have covered the sphere in the last section enable us to describe the position of a point on a sphere in the same way as by the use of graph paper we can describe the position of a point in a plane. In the case of the sphere, the "curves of reference" may be taken to be the equator which we have drawn and one of the meridians which may be called the "prime meridian."

This is the system which is employed on the surface of the earth, the two series being the parallels of latitude and the lines of longitude. The curves of reference are the Equator of the earth and the Meridian through the central "wire" of the Meridian-Circle at Greenwich Observatory.

It is evident that if R be the radius of the sphere and r the

radius of a small circle pq, we have $PQ = R\theta$ and $pq = r\theta$, where θ is the angle between the planes containing NpP and NqQ. Therefore

$$pq = PQ . r/R = PQ \sin p\hat{O}N = PQ \cos \phi .$$

In other words, *distance measured along a parallel of latitude is equal to the difference in longitude × cosine of the latitude.*

§ 7. **Spherical Triangles.**—The straight lines in which the planes of fig. 1 cut each other consist of the axis and other lines

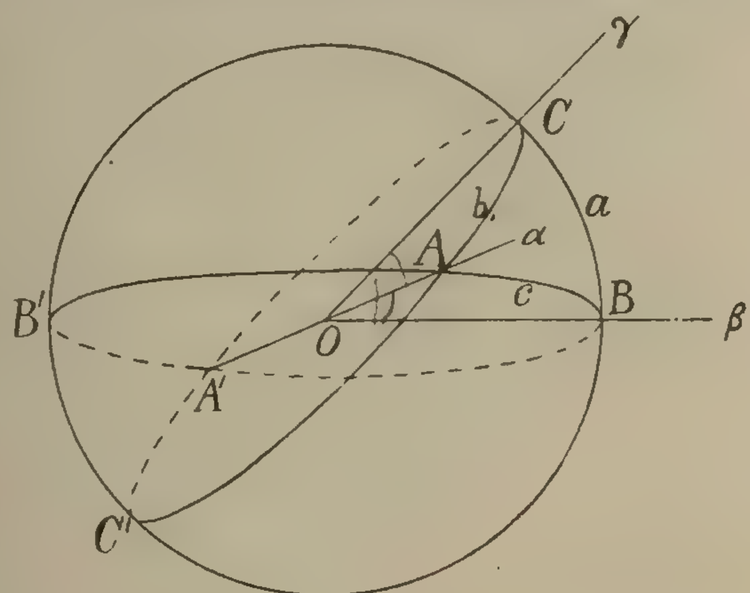

FIG. 2.

all perpendicular to it. We now proceed to the main problem of this book, viz.:—*Given any three concurrent lines in space, to find the relations that connect the angles between them and between their containing planes.*

Let $O\alpha$, $O\beta$, $O\gamma$ be any three lines through O, so that the three planes, whose intersections they are, form a *trihedral angle.*

About O as centre describe a sphere of unit radius, meeting the three lines in A, B, C (fig. 2). The three containing planes BOC, COA, AOB will plainly cut the sphere in great circles, so that the points A, B, C are joined by arcs of great circles.

A three-sided figure drawn on a sphere is called a *Spherical Triangle* if each of its sides is part of a *great circle*; or we may regard a spherical triangle as the intersection of a sphere and a trihedral angle having its vertex at the centre.

§ **8. Ambiguity regarding the Triangle.**—The three vertices A, B, C do not uniquely determine the triangle; for we may proceed from A to B along a great circle by one or other of two arcs, the one greater than a semicircle, the other less. Similarly for the "sides" BC and CA, so that we have $2 \times 2 \times 2$ or 8 possible triangles all having the same vertices. Since, further, for any one of these triangles the remainder of the sphere also forms a triangle the total number becomes 16. In order to define which triangle is meant in view of this latter alternative, we adopt the following convention: *as we proceed along the sides joining the vertices the triangle is on our left.*

§ **9. Definitions.**—Our problem is thus seen to be similar to that of plane trigonometry, having, however, a sphere instead of a plane as fundamental surface. As in plane trigonometry, the letters A, B, C are used to denote angles, and a, b, c the opposite sides of a triangle. Since the sphere has unit radius, a, b, c also measure the angles between the lines $O\alpha$, $O\beta$, $O\gamma$ (fig. 2). By the "angle A" is meant the angle between the tangents at the point A to the two sides b and c, which join C to A and A to B respectively; the angle is on our left as we proceed along the sides turning the corner at A. Since OA is perpendicular to the arcs b and c at this point, the angle A is also a measure of one of the two angles between the planes COA, AOB, the other angle being its supplement. The student should memorise the letters in the order they occur, viz. a, C, b, A, c, B, a, C, . . . etc.

If OA be produced backwards, it will intersect the sphere again in a point A' diametrically opposite to A, the figure $ABA'C$ forming a lune. Two triangles such as ABC and CBA', which together form a lune, are said to be *co-lunar.*

§ **10. Stereographic Projection.**—Owing to the difficulty of visualising in all cases the possible triangles which can be drawn on a sphere, it is convenient to have a method of representing them on a plane. To do this *stereographic projection* may be employed. Imagine a plane P drawn to touch the sphere at *any* point O; and from the diametrically opposite point O' project the sides of the spherical triangle on the plane. Since all circles project into circles and all angles into *equal* angles, it is easy to draw the projection of any triangle. Figs. 3A, 3B, 3C, 3D are representative of the different cases that can arise when the three vertices are given. As we traverse the sides in the direction of the arrows, the "interior" of the triangles, and the angles, are on our left. Each figure thus represents two triangles according to the direction of the arrow. Since the parts of the sphere indefinitely near O' project to an indefinitely great distance, it is only those triangles which do not contain O' which have a projection of finite area. In 3A the vertices are joined by sides all less than 180°. Any side greater than 180° must cut any other side (produced, if necessary) in two points, as was seen in the last section. The angles in 3A are all less or all greater than 180° according to the arrow. In 3B *one* of the sides is greater than 180°, so that it is one of three possible cases. Either one or

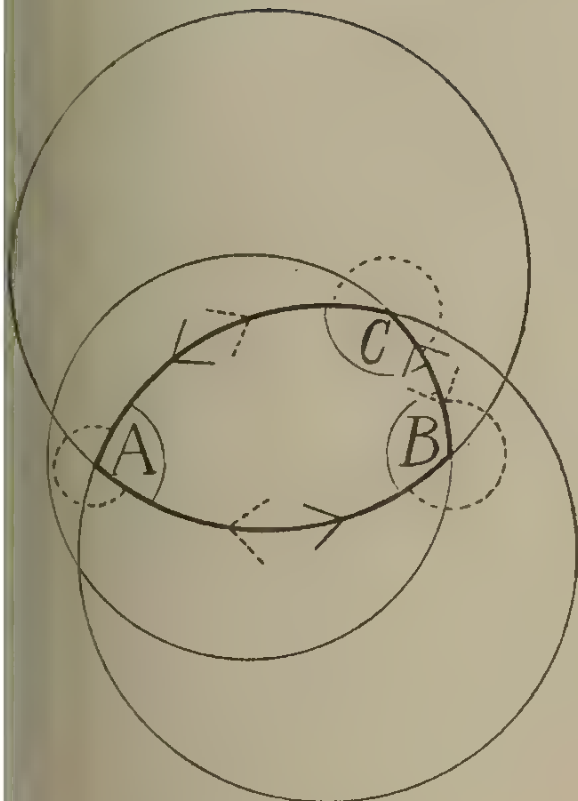

FIG. 3A.

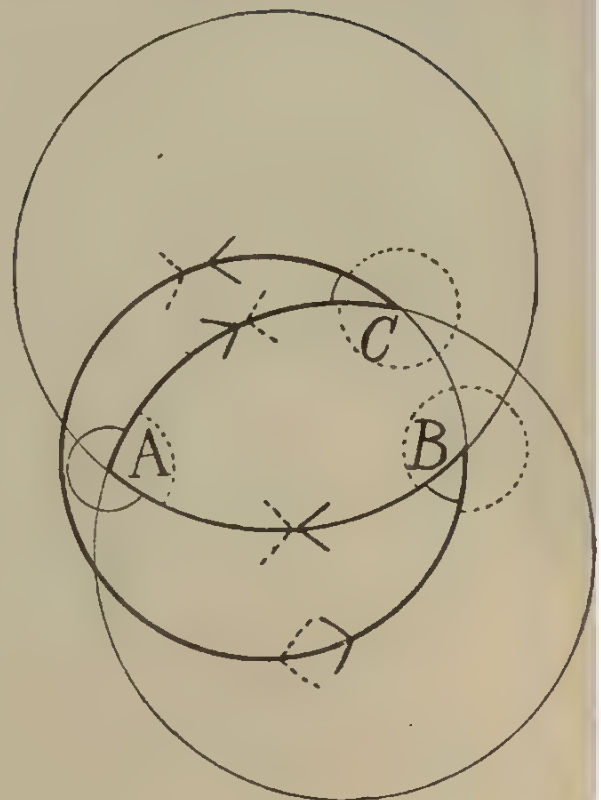

FIG. 3B.

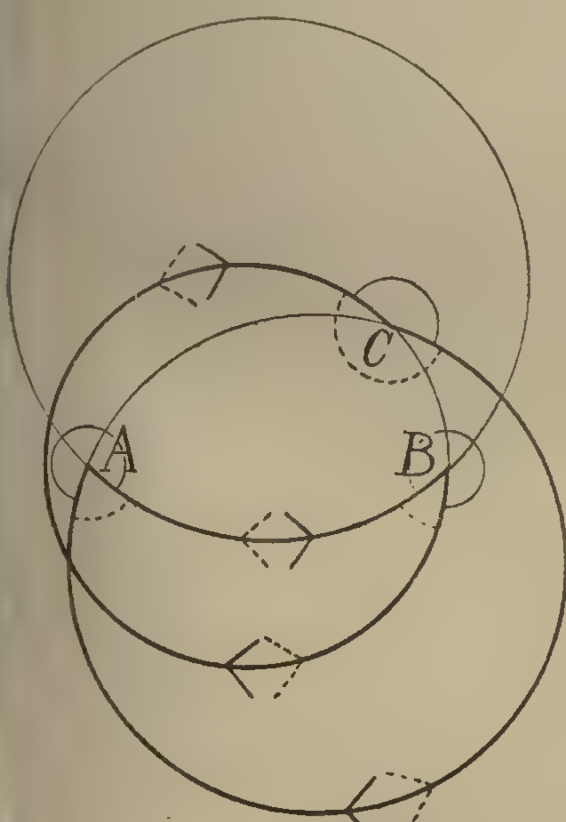

FIG. 3C.

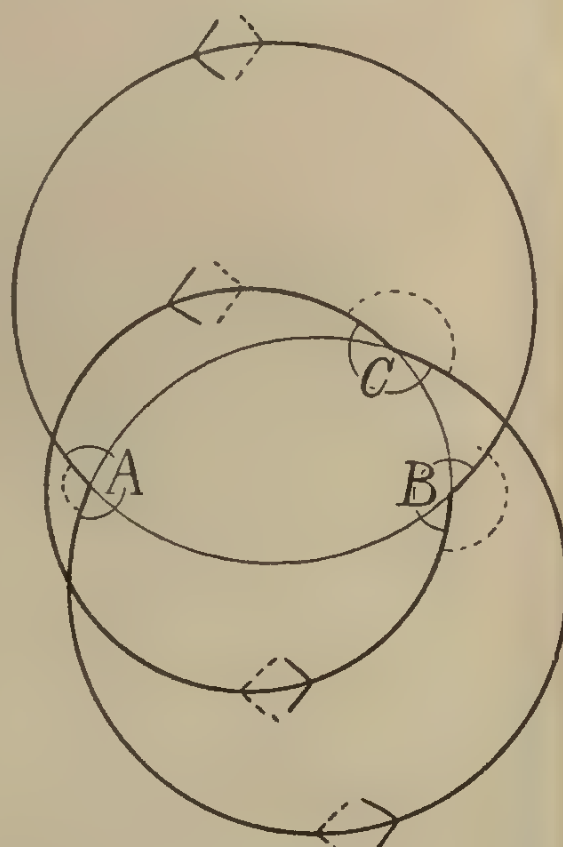

FIG. 3D.

two of the angles are greater than 180° according to the arrow. In 3C *two* of the sides are greater than 180°, so that it also is one of three possible cases, and of the angles one or two are greater than 180° according to the arrow. In 3D *all* the sides are greater than 180° and the angles are all less or all greater than 180° according to the arrow. We have thus accounted for all the eight (or sixteen) possible triangles having given vertices. Since in practical applications no side ever cuts another side in more than one point without being produced, cases 3C and 3D occur only in theoretical questions.

To draw these circles it is convenient to choose as point O the pole of one of the circles a, say, so that the arc a is drawn to scale (the poles of the other circles do not project into centres). Mark the point C' diametrically opposite to C, and through O draw a line OL perpendicular to CC'. With some point on L as centre, draw the circle b through C and C', making the angle at C equal to the given value. Similarly draw the side c to pass through the point B' diametrically opposite B.

§ **11. Possible Triangles.**—It can be shown from the preceding construction, and will be demonstrated later (§ 22), that when any three of the six parts A, c, B, a, C, b are given, there are only two possible triangles (if any). Even this ambiguity is generally removed in practice by the conditions of the problem. For instance, a large class of problems in astronomy restricts one side to be in the first two quadrants, while still oftener each of the parts is required to be less than two right angles, or to have *positive* sines, a restriction which we may shortly refer to as the *Sine Convention.* It will also be shown as we proceed that when one of the possible alternatives for the fourth part is chosen, then the remaining two parts are uniquely determinate. Our problem is thus seen to be equivalent to *finding a fourth part when three are given, i.e.* to finding relations connecting four parts. Following d'Ocagne, we divide the different cases that can arise into three groups:—

§ **12. D'Ocagne's Classification.**—Let us number the consecutive parts of the triangle, viz. a, C, b, A, c, B, a, . . . etc., 1, 2, 3, 4, 5, 6, beginning with any one of them.

The Case (2.2).—The four parts in this case consist of two groups of two separated from each other by one intervening part, *e.g.*, A, c, —, a, C, or by number—say, 1, 2, 4, 5.

The Case (3.1).—Here three of the parts are consecutive and the other separated. We may number them 1, 2, 3, 5, *e.g.*, a, C, b, —, c, or A, c, B, —, C.

The Case (4.0).—Here all four parts are consecutive 1, 2, 3, 4, *e.g.*, a, C, b, A, or A, c, B, a.

Disregarding ambiguities for the moment, it will be seen that we can solve a triangle completely by the use of (3.1), for by one application we have the parts 1, 2, 3, 5; then by a second application the parts 1, 3, 4, 5; and finally 5, 6, 1, 3. Similarly if at least two of the given parts are adjacent, one application of (4.0) gives, 1, 2, 3, 4; a second 2, 3, 4, 5; and finally 3, 4, 5, 6. We cannot, however, solve by the use of (2.2) alone.

We now proceed to derive the *fundamental* formulæ for these three cases.

§ 13. The Case (2.2). The Sine Formula.—Cut off from OA (fig. 4) OP of unit length, and through P draw a plane having OA as normal and cutting OB, OC in Q and R respectively. Let M be the projection on the line PQ of the point R. The angle RPQ is the angle A between the planes COA, AOB; and a, b, c are as shown.

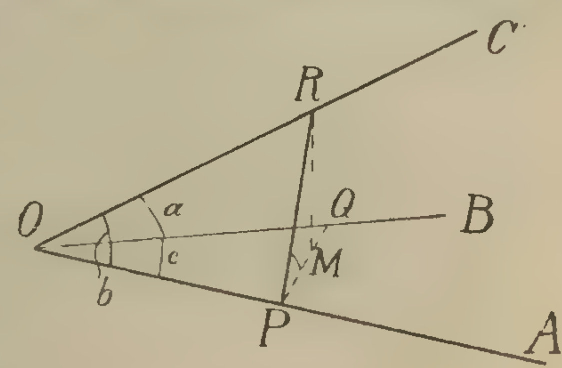

FIG. 4.

We have $MR = PR \sin \mathrm{A} = OR \sin b \sin \mathrm{A}$.

Similarly $MR = OR \sin a \sin \mathrm{B}$.

Hence

Similarly

$$\left.\begin{aligned} \sin a \sin \mathrm{B} &= \sin \mathrm{A} \sin b \\ \sin b \sin \mathrm{C} &= \sin \mathrm{B} \sin c \\ \sin c \sin \mathrm{A} &= \sin \mathrm{C} \sin a \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

these equations, being obtained by projection, are true for all quadrants of the variables involved.

§ 14. Transformations of the Sine Formula.—From

$$\frac{\sin \mathrm{A}}{\sin \mathrm{B}} = \frac{\sin a}{\sin b}$$

we have

$$\frac{\sin \mathrm{A} - \sin \mathrm{B}}{\sin \mathrm{A} + \sin \mathrm{B}} = \frac{\sin a - \sin b}{\sin a + \sin b},$$

i.e.

$$\tan \tfrac{1}{2}(\mathrm{A} - \mathrm{B}) \tan \tfrac{1}{2}(a + b) = \tan \tfrac{1}{2}(a - b) \tan \tfrac{1}{2}(\mathrm{A} + \mathrm{B}) \quad . \quad . \quad (2)$$

If the sine convention holds so that $\frac{1}{2}(a+b)$ and $\frac{1}{2}(\mathrm{A}+\mathrm{B})$ are both in the first quadrant, this shows that $\tan \frac{1}{2}(\mathrm{A} - \mathrm{B})$ and $\tan \frac{1}{2}(a - b)$ are of the same sign, *i.e.* $\mathrm{A} - \mathrm{B}$ and $a - b$ are either both positive or both negative.

Similarly we have

$$\tan \tfrac{1}{2}(A-a) \tan \tfrac{1}{2}(B+b) = \tan \tfrac{1}{2}(B-b) \tan \tfrac{1}{2}(A+a) \qquad . \quad . \quad (3)$$

which shows that with the same convention $A-a$ and $B-b$ are also both positive or both negative.

The sine formula can also be written in the form

$$\sin a \sin^2(45^\circ - \tfrac{1}{2}B) = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}(a-b) + \sin b \sin^2(45^\circ - \tfrac{1}{2}A) \quad . \quad (4)$$

§ 15. **The Case (3.1). The Cosine Formula.**—We shall now find the relation connecting three consecutive parts of a spherical triangle and a separated side.

Referring again to fig. 4, we have, since P is the projection of both R and Q on OA—

$$OR = \sec b \qquad PR = \tan b$$
$$OQ = \sec c \qquad PQ = \tan c.$$

Applying plane trigonometry to the triangles ORQ and PRQ, we have

$$RQ^2 = OR^2 + OQ^2 - 2OR \,.\, OQ \cos a$$

and

$$RQ^2 = PR^2 + PQ^2 - 2PR \,.\, PQ \cos A$$

i.e.

$$RQ^2 = \sec^2 b + \sec^2 c - 2 \sec b \sec c \cos a$$

and

$$RQ^2 = \tan^2 b + \tan^2 c - 2 \tan b \tan c \cos A.$$

∴ Subtracting, we have

$$0 = 1 + \tan b \tan c \cos A - \sec b \sec c \cos a,$$

or, multiplying throughout by $\cos b \cos c$ and rearranging,

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \text{Similarly} \qquad \cos b &= \cos c \cos a + \sin c \sin a \cos B \\ \text{and} \qquad \cos c &= \cos a \cos b + \sin a \sin b \cos C \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

To complete the solution of the case (3.1) we require the relation connecting three consecutive parts and a separated *angle*, *e.g.* A, c, B, and C. This will be given in § 21.

§ 16. **Another Proof of the Sine Formula.**—The Cosine formula is often called the *fundamental formula* of Spherical Trigonometry from the fact that if none of the parts exceed 180°, all the other formulæ may be deduced from it. The Sine formula can be deduced as follows :—

From
$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$
we have
$$\sin^2 A = 1 - \cos^2 A = \frac{(1-\cos^2 b)(1-\cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$
$\therefore$
$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}$$
$\therefore$ by symmetry
$$\frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c};$$
assuming all the sines positive, and taking roots, we obtain the Sine formula.

§ **17. Auxiliary Angles.**—This cosine formula is not, as is the sine formula, in a form adapted to logarithmic calculation. If, however, we introduce a new quantity θ defined by
$$\tan\theta = \tan b \cos A,$$
the first equation of group (5) can be written
$$\cos a = \cos b \cos c + \sin b \sin c \cot b \tan\theta$$
i.e.
$$\cos\theta \cos a = \cos(c-\theta)\cos b \quad . \quad . \quad . \quad (\alpha)$$
which is in the required product form.

Similarly, interchanging b and c, we have
$$\tan\theta' = \tan c \cos A$$
$$\cos\theta' \cos a = \cos(b-\theta')\cos c \quad . \quad . \quad . \quad (\beta)$$
Having first found θ (or θ'), supposing that A and b (or c) are given, we obtain the required fourth part a or c (or b) from equation (α) or (β).

Similarly for any other equation of group (5).

Example 1.—Given $b = 49°\ 24'\ 10''\cdot 23$, $A = 110°\ 51'\ 14''\cdot 84$, $a = 70°\ 14'\ 20''\cdot 12$, find $c < 180°$.

$$\begin{aligned}
\text{L}\tan b &= 10\cdot 0670\ 102{,}5 \\
\text{L}\cos A &= \underline{9\cdot 5514\ 376{,}1n} \\
\text{L}\tan\theta &= 9\cdot 6184\ 478{,}6n \qquad \theta = -22°\ 33'\ 25''\cdot 66 \\
\text{L}\cos b &= \underline{9\cdot 8134\ 052{,}1} \\
-\text{L}\cos b &= 0\cdot 1865\ 947{,}9 \\
\text{L}\cos a &= 9\cdot 5290\ 434{,}7 \\
\text{L}\cos\theta &= \underline{9\cdot 9654\ 357{,}2} \\
\text{L}\cos(c-\theta) &= 9\cdot 6810\ 739{,}8 \qquad c-\theta = \underline{61°\ 19'\ 36''\cdot 01} \\
& \qquad\qquad\qquad\qquad\quad c = 38°\ 46'\ 10''\cdot 35.
\end{aligned}$$

The student should verify the above calculation as far as seven-place tables will permit by using the value of c obtained, together with a and A, to find b.

Example 2.—Given $a = 31°\ 42'\ 16''\cdot 4$, $b = 36°\ 17'\ 10''\cdot 0$, $C = 98°\ 12'\ 40''\cdot 0$, find c.
(*Ans.*: $50°\ 6'\ 24''\cdot 0$.)

§ 18. **Employment of Gaussian Logarithms.**—By means of the "auxiliary angle" θ of last section, we can find the value of any one of the four variables of the cosine formula when the other three are given. When, however, the cosine of the part to be found from this formula can be expressed directly as the sum of two terms both calculable, we can proceed directly and with slightly less working by means of the Gaussian logarithms of § 4. For example, we have

$$\cos a = (\cos b \cos c) + (\sin b \sin c \cos A)$$

and

$$\cos A = \frac{\cos a}{\sin b \sin c} + (-\cot b \cot c).$$

Taking the value for c obtained in the example worked above, viz. 38° 46′ 10″·35, together with the given values of b and A, we can solve in this way for a. The calculation has been performed in § 4, giving $a = 70° \; 14' \; 20''\cdot12$, and thus verifying the value obtained for c.

Examples.—Given $b = 49° \; 24' \; 10''\cdot23$, $c = 38° \; 36' \; 10''\cdot35$, show that the following values for a and A correspond to each other:—

$$\begin{array}{ll} a_1 = 70° \; 14' \; 20''\cdot12 & A_1 = 110° \; 51' \; 14''\cdot84 \\ a_2 = 71° \; 10' \; 21''\cdot00 & A_2 = 112° \; 51' \; 3''\cdot16 \\ a_3 = 82° \; 27' \; 43''\cdot00 & A_3 = 139° \; 1' \; 16''\cdot75 \\ a_4 = 310° \; 15' \; 19''\cdot00 & A_4 = 73° \; 0' \; 55''\cdot89. \end{array}$$

§ 19. **Case (4.0). Cotangent Formulæ.**—If in the equation

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

we substitute for $\cos c$ and $\sin c$ from

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

and

$$\sin c = \sin C \sin a / \sin A$$

we obtain

$$\cot a \sin b = \cos b \cos C + \cot A \sin C,$$

connecting the four consecutive parts a, C, b, A.

By interchanging letters we can write down five similar equations, or altogether the following six:—

$$\left.\begin{array}{l} \cot a \sin b = \cos b \cos C + \cot A \sin C \\ \cot b \sin c = \cos c \cos A + \cot B \sin A \\ \cot c \sin a = \cos a \cos B + \cot C \sin B \\ \cot b \sin a = \cos a \cos C + \cot B \sin C \\ \cot c \sin b = \cos b \cos A + \cot C \sin A \\ \cot a \sin c = \cos c \cos B + \cot A \sin B \end{array}\right\} \quad . \quad . \quad . \quad (6)$$

These, as will be seen, solve all possible cases of four consecutive parts, and are all included in the following rule:—

If we number any four consecutive parts **1**, **2**, **3**, **4** either way round where **1** is a *side*, then

$$\cos \mathbf{2} \cos \mathbf{3} = \begin{vmatrix} \cot \mathbf{1} & \cot \mathbf{4} \\ \sin \mathbf{2} & \sin \mathbf{3} \end{vmatrix}$$

It will be noticed that the cotangents are for the first and last parts, and that the numbers go in regular order round the determinant.

§ **20. Solution by Gaussian Logarithms.**—In each of the above cotangent group of formulæ, two of the variables (b and C in the first one) enter twice, while the remaining two (a, A), which are the first and last ones of the four consecutive parts, occur once only. It follows that, just as was the case with the cosine group, we can use Gaussian logarithms (§ 18) to solve for either of the two end parts in the case (4.0). It will be shown later that any one of the four parts can be found directly when three are given, by introducing an auxiliary angle into the cotangent formula.

Examples:—

(1) Given $b=49° \; 24' \; 10''{\cdot}23$, $A=110° \; 51' \; 14''{\cdot}84$, $c=38° \; 46' \; 10''{\cdot}35$, find in succession by the cotangent formula B and a (less than 180°).

($B=48° \; 56' \; 5''{\cdot}07$; $a=70° \; 14' \; 20''{\cdot}12$.)

(2) Given $a=21° \; 32' \; 0''{\cdot}50$, $C=171° \; 48' \; 42''{\cdot}40$, $b=23° \; 27' \; 25''{\cdot}53$, find A and c ($<180°$). ($A=4° \; 14' \; 58''{\cdot}00$; $c=44° \; 52' \; 11''{\cdot}00$.)

§ **21. Completion of Cosine Formula.**—Applying the cotangent formula to the first four of B, a, C, b, A, and to the last four, we have

$$\cot B \sin C - \cot b \sin a = -\cos a \cos C \quad . \quad . \quad . \quad (1')$$

and

$$\cot A \sin C - \cot a \sin b = -\cos b \cos C \quad . \quad . \quad . \quad (2')$$

or, since $\sin a/\sin b = \sin A/\sin B$,

$$\cot B \sin C - \cos b \sin A/\sin B + \cos a \cos C = 0 \quad . \quad . \quad (3')$$

and

$$\cot A \sin C - \cos a \sin B/\sin A + \cos b \cos C = 0 \quad . \quad . \quad (4')$$

To eliminate the fourth part b multiply (3′) by $\sin B \cos C$ and (4′) by $\sin A$ and add; there results after division by $\sin C$—

Similarly and

$$\left.\begin{aligned}\cos A &= -\cos B \cos C + \sin B \sin C \cos a \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c\end{aligned}\right\} \quad . \quad . \quad (7)$$

These three equations together with group (5) complete the case (3.1).

§ 22. **Fundamental Formulæ.**—We may now group together for reference representatives of the standard formulæ for the three different cases—

Sine Formula. Case (2.2).

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad . \quad . \quad . \quad . \quad (\alpha)$$

Cosine Formula. Case (3.1).

$$\left.\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \cos A &= -\cos B \cos C + \sin B \sin C \cos a\end{aligned}\right\} \quad . \quad . \quad (\beta)$$

Cotangent Formula. Case (4.0).

$$\cot a \sin b = \cot A \sin C + \cos b \cos C \quad . \quad . \quad . \quad (\gamma)$$

It is to be kept in mind that any one of these groups can be deduced from the other two, so that there are really only two *fundamental* formulæ.

If we are given any three of the six parts A, c, B, a, C, b, we can, by means of (α), or (β), or (γ), obtain the value of the sine, cosine, or tangent of any fourth part. This restricts the part to be in one of two quadrants, but does not solve uniquely. There are thus always two possible values when three parts are given, unless one of them is excluded by other considerations. Having decided upon one of the alternatives, then, to get a fifth part we can employ two of the three forms (α), (β), (γ)—a third would give no further information—and obtain values for the sine and cosine, the sine and tangent, or the cosine and tangent as the case may be, and thus the quadrant as well as the magnitude of the part is determined. Similarly for the sixth part. We must therefore keep in mind that unless the problem is otherwise limited we need two of the equations a, β, γ—generally either the cosine or the cotangent formula is chosen together with the sine formula—to solve the triangle.

Example.—Given $b=50°\ 6'\ 24''\cdot 0$, $A=49°\ 46'\ 10''\cdot 1$, $C=42°\ 41'\ 1''\cdot 8$, find a from the cotangent formula. (*Ans.*: $a=36°\ 17'\ 10''\cdot 0$.)

§ **23. A Transformation of the Formulæ.**—It will be noticed that the substitution of $\pi-A$, $\pi-B$, $\pi-C$ for A, B, C respectively in the two sets of formulæ (5) and (7) makes (5) identical with (7) except that where one has angles the other has sides. The same substitution affects the six cotangent formulæ in the same way if we divide them arbitrarily into two groups of three. Similarly again for the sine formulæ. We thus see that if for angles we took in all cases their supplements, we would always be at liberty, in any formula to be deduced, to interchange sides and angles. Such a convention as regards angles would be equivalent to the following: Instead of taking A as usual to mean the angle inside the triangle—on the left—as we traverse its side in the direction of the arrows (fig. 5), we take it to mean the angle between the positive directions of the sides b and c at the point A.

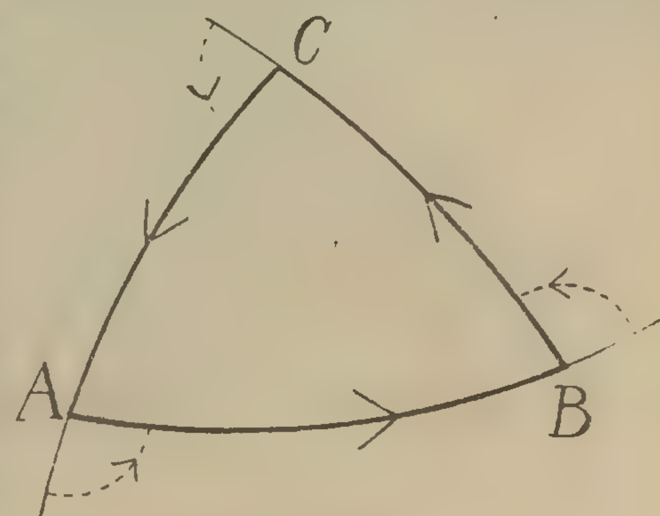

Fig. 5.

Although it would be an advantage in certain cases to be able to interchange sides and angles in this simple manner, the convention has not been adopted practically.

24. The Polar Substitution.—We notice further that if we change sides into angles and angles into sides in the different cases α, β, γ, by the substitution

$$\begin{array}{lll} A=\pi-a' & B=\pi-b' & C=\pi-c' \\ a=\pi-A' & b=\pi-B' & c=\pi-C' \end{array}$$

and then omit the accents, we merely obtain other formulæ of the same group over again. This proceeding is therefore valid, and in any formula which we deduce from α, β, γ we shall be at

liberty to make this substitution. We shall refer to it as the *polar substitution.*

§ 25. Half-Angles in Terms of Sides.—We now proceed to deduce other useful formulæ from the fundamental ones.

We have

$$2\cos^2 \tfrac{1}{2}\mathrm{A} = 1 + \cos \mathrm{A} = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a - \cos (b+c)}{\sin b \sin c} = \frac{2 \sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(b+c-a)}{\sin b \sin c}$$

$$= \frac{2 \sin s \sin (s-a)}{\sin b \sin c} \quad \text{where} \quad 2s = a+b+c.$$

Therefore

$$\cos \tfrac{1}{2}\mathrm{A} = \pm \sqrt{\left\{ \frac{\sin s \sin (s-a)}{\sin b \sin c} \right\}} \quad . \quad . \quad . \quad (8)$$

and two similar formulæ for $\cos \frac{1}{2}\mathrm{B}$ and $\cos \frac{1}{2}\mathrm{C}$.

Again,

$$2 \sin^2 \tfrac{1}{2}\mathrm{A} = 1 - \cos \mathrm{A} = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

whence

$$\sin \tfrac{1}{2}\mathrm{A} = \pm \sqrt{\left\{ \frac{\sin (s-b) \sin (s-c)}{\sin b \sin c} \right\}} \quad . \quad . \quad (9)$$

and two similar formulæ for $\sin \frac{1}{2}\mathrm{B}$ and $\sin \frac{1}{2}\mathrm{C}$.

By division we have

$$\tan \tfrac{1}{2}\mathrm{A} = \pm \sqrt{\left\{ \frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)} \right\}} \quad . \quad . \quad (10)$$

with two similar formulæ for $\tan \frac{1}{2}\mathrm{B}$ and $\tan \frac{1}{2}\mathrm{C}$, where in every case $2s = a+b+c$.

If we write

$$m = + \sqrt{\{\sin (s-a) \sin (s-b) \sin (s-c)/\sin s\}},$$

equation (10) can be conveniently written

$$\left.\begin{aligned} &\tan \tfrac{1}{2}\mathrm{A} = \pm \frac{m}{\sin (s-a)} \\ \text{Similarly} \qquad &\tan \tfrac{1}{2}\mathrm{B} = \pm \frac{m}{\sin (s-b)} \\ \text{and} \qquad &\tan \tfrac{1}{2}\mathrm{C} = \pm \frac{m}{\sin (s-c)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

In nearly all practical cases the plus sign is to be taken before the roots in the above formulæ, and it will be shown in § 28 below that the signs in group (11) are either all plus or all minus.

§ **26. Half-Sides in terms of Angles.**—Proceeding in exactly the same way as in last section with the equation

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

we can show that

$$\cos \tfrac{1}{2}a = \pm\sqrt{\left\{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}\right\}} \quad . \quad . \quad (12)$$

Similarly

$$\sin \tfrac{1}{2}a = \pm\sqrt{\left\{\frac{-\cos S \cos(S-A)}{\sin B \sin C}\right\}} \quad . \quad . \quad . \quad (13)$$

and

$$\tan \tfrac{1}{2}a = \pm\sqrt{\left\{\frac{-\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}\right\}} \quad . \quad . \quad (14)$$

with similar formulæ for $\sin \frac{1}{2}b$, $\sin \frac{1}{2}c$, $\cos \frac{1}{2}b$, $\cos \frac{1}{2}c$, $\tan \frac{1}{2}b$, and $\tan \frac{1}{2}c$, where in all cases $2S = A + B + C$.

Writing

$$M = +\sqrt{\left\{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}\right\}}$$

equation (14) can be conveniently written

$$\left.\begin{aligned} &\tan \tfrac{1}{2}a = \pm M \cos(S-A) \\ \text{Similarly}\quad &\tan \tfrac{1}{2}b = \pm M \cos(S-B) \\ \text{and}\quad &\tan \tfrac{1}{2}c = \pm M \cos(S-C) \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

In nearly all practical cases the plus sign is to be taken before the roots in these formulæ, and it will be shown in § 28 below that the signs in group (15) are either all plus or all minus.

Example.—If p denote the perpendicular from A on the side BC, show that

$$\sin p = \frac{2}{\sin a}\sqrt{\left\{\sin s \sin(s-a)\sin(s-b)\sin(s-c)\right\}}.$$

§ **27. Equations connecting more than Four Parts.**—All the formulæ given above have been for four parts only. Formulæ connecting five or six parts are sometimes useful for solving triangles, but more frequently they serve as convenient check

equations for the values of two or more parts obtained by other means.

Five Parts. Napier's Analogies.—To connect the five parts B, a, C, b, A we combine the cosine formulæ for the parts B, a, C, —, A, and for B, —, C, b, A. Thus—

$$\cos B + \cos C \cos A = \sin C \sin A \cos b$$
$$\cos A + \cos B \cos C = \sin B \sin C \cos a\,;$$

∴ by addition

$$(\cos A + \cos B)(1 + \cos C) = \sin C(\sin A \cos b + \sin B \cos a).$$

Now, since

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = p \text{ (say)},$$

this becomes

$$(\cos A + \cos B)(1 + \cos C) = p \sin C \sin (a + b).$$

But

$$\sin A + \sin B = p\,(\sin a + \sin b)\,;$$

∴ by division

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin a + \sin b}{\sin (a + b)} \quad \frac{1 + \cos C}{\sin C}$$

that is,

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}C \quad . \quad . \quad . \quad (16)$$

Now, by equation (3)

$$\tan \tfrac{1}{2}(A - B) \tan \tfrac{1}{2}(a + b) = \tan \tfrac{1}{2}(a - b) \tan \tfrac{1}{2}(A + B)\,;$$

multiplying both sides of (16) by the corresponding sides of this equation we have—

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2}C \quad . \quad . \quad . \quad (17)$$

Similarly we can derive equations connecting the five parts B, a, C, b, A; or, applying the polar substitution to (16) and (17), we have at once

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \tfrac{1}{2}c \quad . \quad . \quad . \quad (18)$$

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \tfrac{1}{2}c \quad . \quad . \quad . \quad (19)$$

The four equations (16), (17), (18), (19) were discovered by *Napier* and published in *1614* in his *Mirifici Logarithmorum Canonis Descriptio.* The name *Analogy* refers to the form in which they were originally given, as proportions

$$\cos \tfrac{1}{2}(a + b) : \cos \tfrac{1}{2}(a - b) : : \cot \tfrac{1}{2}C : \tan \tfrac{1}{2}(A + B), \text{ etc.}$$

Example.—For a right-angled triangle ($A=90°$) establish Prony's Theorem

$$\sin(a-c)=\sin b.\cos c.\tan\tfrac{1}{2}B=\tan b.\cos a.\tan\tfrac{1}{2}B.$$

§ **28. Ambiguous Signs in Tangent Formulæ.**—In group (11), expressing the tangent of half-angles in terms of the sides, the signs are to be taken either all plus or all minus in any triangle.

For, if possible, let $\tan\frac{1}{2}A$ have a plus sign before the root and $\tan\frac{1}{2}B$ a minus sign. Then, by division—

$$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}B}=-\frac{\sin(s-b)}{\sin(s-a)}$$

$$\therefore\ \frac{\tan\frac{1}{2}A-\tan\frac{1}{2}B}{\tan\frac{1}{2}A+\tan\frac{1}{2}B}=\frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)}$$

$$i.e.\quad \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)}$$

This by (19) implies that $\sin^2\frac{1}{2}(A-B)=\sin^2\frac{1}{2}(A+B)$, which cannot be true in general; whence the result.

In exactly the same way, making use of the corresponding Napier Analogy (17), we can show that in group (15) the signs are either all plus or all minus.

§ **29. All the Parts. Delambre's Analogies.**—By the preceding section

$$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}B}=\frac{\sin\frac{1}{2}A\cos\frac{1}{2}B}{\cos\frac{1}{2}A\sin\frac{1}{2}B}=+\frac{\sin(s-b)}{\sin(s-a)}\quad . \quad . \quad . \quad (1')$$

Adding unity to both sides, we have

$$\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}A\sin\frac{1}{2}B}=\frac{2\sin\frac{1}{2}c\cos\frac{1}{2}(a-b)}{\sin(s-a)}\quad . \quad . \quad . \quad (2')$$

Now, by direct substitution of the formulæ in § 25, we easily find

$$\frac{\cos\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}C}=\pm\frac{\sin(s-a)}{\sin c}.\quad . \quad . \quad (3')$$

Multiplying corresponding sides of (2′) and (3′), we have

$$\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}C}=\pm\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\quad . \quad . \quad . \quad (20)$$

Again, subtracting unity from both sides of (1′), we have

$$\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}A\sin\frac{1}{2}B}=\frac{2\cos\frac{1}{2}c\sin\frac{1}{2}(a-b)}{\sin(s-a)},$$

and multiplying corresponding sides of this and (3′),

$$\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}=\pm\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}\quad . \quad . \quad . \quad (21)$$

The doubtful signs in (20) and (21) are either both plus or both minus in any triangle; for division of corresponding sides gives agreement with (19) only on this condition.

Applying the polar substitution to (20), we have

$$\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C} = \pm\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c} \quad . \quad . \quad . \quad (22)$$

the upper signs or the lower signs of the two equations being taken together.

Multiplying this by the corresponding sides of (18) written in the form

$$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} = \frac{\cot\frac{1}{2}(a+b)}{\cot\frac{1}{2}c},$$

we have

$$\frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}C} = \pm\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c} \quad . \quad . \quad . \quad (23)$$

with the same rule as regards double sign.

Equations (20), (21), (22), (23) thus form a group in which for any triangle the signs are either all positive or all negative.

They were discovered by *Delambre* (1809).

Example.—Verify Delambre's analogies numerically for the spherical triangle whose parts are

$$\begin{array}{ll} a=50^\circ\ \ 6'\ 24''\cdot0 & A=98^\circ\ 12'\ 40''\cdot0 \\ b=31^\circ\ 42'\ 16''\cdot4 & B=42^\circ\ 41'\ \ 1''\cdot8 \\ c=36^\circ\ 17'\ 10''\cdot0 & C=49^\circ\ 46'\ 10''\cdot1. \end{array}$$

§ 30. **Areas of Spherical Triangles.**—From our definition of a *lune* (§ 4) it is obvious that its area is the same fraction of the area of the sphere as its angle is of four right angles, *i.e.*

$$\textit{Area of Lune} = (A/2\pi)\times 4\pi r^2 = 2Ar^2,$$

where A is the angle of the lune and r is the radius of the sphere.

Referring to fig. 2, we may regard the hemisphere on which the triangle ABC lies as equivalent to the lune $C'BCA$ + the lune $B'CBA$ + (the triangle ABC + the triangle $AB'C'$) − twice the triangle ABC. Now, it is evident that the triangles ABC and $A'B'C'$ are equal in area, each element of the one being diametrically opposite to a corresponding one of the other, so that the triangles ABC and $AB'C'$ together form a lune of angle A.

Therefore remembering the expression for the area of a lune, and denoting the required area of ABC by Δ, we have

$$\textit{Area of hemisphere} = 2\pi r^2 = 2\mathrm{A}r^2 + 2\mathrm{B}r^2 + 2\mathrm{C}r^2 - 2\Delta, \text{ or}$$
$$\textit{Area of spherical triangle} = (\mathrm{A}+\mathrm{B}+\mathrm{C}-2\pi)r^2.$$

The expression $\mathrm{A}+\mathrm{B}+\mathrm{C}-2\pi$, denoting the amount by which the sum of the angles of a spherical triangle exceeds those of a plane triangle, is called the *Spherical Excess* and is frequently denoted by the letter E. Evidently in the notation of § 26 we have

$$\mathrm{S} = \tfrac{1}{2}\pi + \tfrac{1}{2}\mathrm{E}.$$

Example 1.—Deduce from Delambre's Analogies the formulæ

$$\tan\tfrac{1}{4}\mathrm{E}\tan\tfrac{1}{4}(2\mathrm{A}-\mathrm{E}) = \tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)$$
$$\tan\tfrac{1}{4}\mathrm{E}\cot\tfrac{1}{4}(2\mathrm{A}-\mathrm{E}) = \tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a),$$

and hence derive *L'Huilier's Theorem*

$$\tan\tfrac{1}{4}\mathrm{E} = \{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)\}^{\frac{1}{2}}.$$

Example 2.—Assuming the earth to be a perfect sphere of 4000 miles radius, find the area of a right-angled isosceles triangle drawn upon it whose equal sides are each 100 miles long. (*Ans.* : 5058 square miles.)

Example 3.—The excess of the three angles of a triangle measured on the earth's surface, above two right angles, is one second: what is its area, taking the earth's diameter at 8000 miles? (*Ans.* : 77·57 square miles.)

CHAPTER III

THE NUMERICAL SOLUTION OF THE RIGHT-ANGLED TRIANGLE

§ 31. The Various Cases and their Solution.—The solution of triangles in which one angle is a right angle acquires importance from the fact, which will appear later, that a large part of the numerical calculation of sides and angles in any triangle depends essentially on breaking it up into two right-angled triangles.

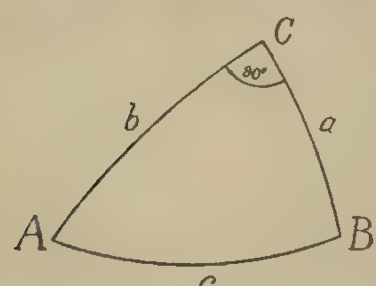

FIG. 6.

The angle (C, say) being given equal to a right angle, we need only two other parts in order to be able to solve the triangle. The different cases, with the formulæ which are required to furnish the unknown parts in each case, are given in the following table:—

Parts given.	*Parts required.*
I. Two sides a and b.	$\cos c = \cos a \cos b$ $\cot \mathrm{A} = \cot a \sin b$ $\cot \mathrm{B} = \cot b \sin a$
II. One side a and the hypotenuse c.	$\cos b = \cos c/\cos a$ $\cos \mathrm{B} = \tan a \cot c$ $\sin \mathrm{A} = \sin a/\sin c$
III. One side a and the opposite angle A.	$\sin c = \sin a/\sin \mathrm{A}$ $\sin b = \tan a \cot \mathrm{A}$ $\sin \mathrm{B} = \cos \mathrm{A}/\cos a$

Case III. is known as the *ambiguous case*, from the fact that the given data are compatible with two distinct relative configurations of the angular points on the sphere. If one of these be ABC, the other may be represented by A′BC, where A′ is the antipodes of A.

Parts given.	*Parts required.*
IV. One side b and the adjacent angle A.	$\tan c = \tan b/\cos A$ $\tan a = \tan A \sin b$ $\cos B = \cos b \sin A$
V. The hypotenuse c and one angle A.	$\tan b = \tan c \cos A$ $\cot B = \cos c \tan A$ $\sin a = \sin c \sin A$
VI. The two angles A and B.	$\cos c = \cot A \cot B$ $\cos a = \cos A/\sin B$ $\cos b = \cos B/\sin A$

Example 1.—Given $A = 80°\ 10'\ 30''{\cdot}10$, $c = 110°\ 46'\ 20''{\cdot}12$, $C = 90°$, find a and b ($b < 180°$).

$\sin a = \sin A \sin c$

$$\begin{aligned} L \sin c &= 9{\cdot}9708\ 104{,}8 \\ L \sin A &= \underline{9{\cdot}9935\ 833{,}1} \\ L \sin a &= 9{\cdot}9643\ 937{,}9 \\ a &= 67°\ 6'\ 52''{\cdot}5(6) \end{aligned}$$

$\tan b = \tan c \cos A$

$$\begin{aligned} L \tan c &= 0{\cdot}4210\ 053{,}1n \\ L \cos A &= \underline{9{\cdot}2320\ 782{,}1} \\ L \tan b &= 9{\cdot}6530\ 835{,}2n \\ b &= -24°\ 13'\ 16''{\cdot}96 \\ &= 155°\ 46'\ 43''{\cdot}04. \end{aligned}$$

Check: $\cos c = \cos a \cos b$

$$\begin{aligned} L \cos a &= 9{\cdot}5898\ 259{,}0 \\ L \cos b &= \underline{9{\cdot}9599\ 791{,}6n} \\ &\ 9{\cdot}5498\ 050{,}6n \end{aligned} \qquad \begin{aligned} L \sin c &= 9{\cdot}9708\ 104{,}8 \\ L \tan c &= \underline{0{\cdot}4210\ 053{,}1n} \\ &\ 9{\cdot}5498\ 051{,}7n. \end{aligned}$$

Example 2.—With the same data, show that $B = 153°\ 58'\ 24''{\cdot}98$.

Example 3.—Given $a = 49°\ 24'\ 10''{\cdot}23$, $c = 69°\ 8'\ 45''{\cdot}16$, $A = 90°$, find $b < 180°$.
(*Ans.:* $b = 22°\ 33'\ 25''{\cdot}66$.)

Example 4.—Given $C = 90°$, $a = 50°\ 10'\ 11''{\cdot}10$, $b = 210°\ 42'\ 15''{\cdot}18$, find A and B ($B > 180°$).
(*Ans.:* $A = 113°\ 4'\ 5''{\cdot}12$
$B = 217°\ 42'\ 53''{\cdot}86$.)

Example 5.—Given $a = 23°\ 39'\ 17''{\cdot}0$, $b = 18°\ 51'\ 21''{\cdot}0$, $C = 90°$, find B.
(*Ans.:* $B = 40°\ 24'\ 14''{\cdot}05$.)

Example 6.—Given $A = 53°\ 22'\ 22''{\cdot}35$, $a = 28°\ 21'\ 23''{\cdot}0$, $C = 90°$, show that one of the values of B is $42°\ 41'\ 1''{\cdot}8$.

Example 7.—Given $A = 44°\ 50'\ 17''\ 7$, $c = 31°\ 42'\ 16''{\cdot}4$, $C = 90°$, find a.
(*Ans:* $a = 21°\ 45'\ 1''{\cdot}0$.)

§ 32. **Cases requiring Special Treatment.**—If a part happens to have a value near $\pi/2$ or $3\pi/2$ and is to be determined by its sine, or to be near $0°$ or π and is to be determined by its cosine, the tables we are using may not give it with sufficient accuracy.

In such cases the difficulty can usually be got over by first finding some other part which can be well determined and using it with one of the given parts to determine the required part. For example, in Case V. the formula $\sin a = \sin c \sin A$ gives a poor determination of a if it is nearly 90°. We have, however, b well determined by $\tan b = \tan c \cos A$, after which a is given by $\tan a = \sin b \tan A$.

It may even happen in Cases II. and III. where all the unknown parts are expressed as sines or cosines that none of them can be well determined. It is then necessary to find expressions for the half-angles, and these take the following form:—

$$\text{Case II.} \left\{ \begin{array}{l} \tan \tfrac{1}{2}b = \pm \sqrt{\{\tan \tfrac{1}{2}(c+a) \tan \tfrac{1}{2}(c-a)\}} \\ \tan \tfrac{1}{2}B = \pm \sqrt{\{\sin (c-a)/\sin (c+a)\}} \\ \tan \tfrac{1}{2}(A+90) = \pm \sqrt{\{\tan \tfrac{1}{2}(c+a)/\tan \tfrac{1}{2}(c-a)\}} \end{array} \right.$$

$$\text{Case III.} \left\{ \begin{array}{l} \tan \tfrac{1}{2}(90-c) = \pm \sqrt{\{\tan \tfrac{1}{2}(A-a)/\tan \tfrac{1}{2}(A+a)\}} \\ \tan \tfrac{1}{2}(90-b) = \pm \sqrt{\{\sin (A-a)/\sin (A+a)\}} \\ \tan \tfrac{1}{2}(90-B) = \pm \sqrt{\{\tan \tfrac{1}{2}(A-a) \tan \tfrac{1}{2}(A+a)\}}. \end{array} \right.$$

§ 33. **Heegmann's Tables.**—It is easily seen that any three parts of a right-angled triangle are connected by one or other of the formulæ

$$\sin z = \sin x \sin y$$
$$\tan z = \tan x \sin y,$$

where x, y, z, are severally the parts themselves or their complements. Making use of this property, A. Heegmann constructed * in 1849 two double-entry tables (one for each formula) for the solution of all right-angled triangles. On entering with the values of the two given parts (or their complements) in the appropriate table, any required part is obtained by interpolation to an accuracy of a tenth of a minute. These tables can also be used for the solution of triangles which are not right-angled by breaking them up into two right-angled ones.

* *Mém. Soc. Sci.*, Lille, 1849, pp. 487–676.

CHAPTER IV

THE NUMERICAL SOLUTION OF THE GENERAL TRIANGLE

§ 34. **Classification of Cases.**—We now proceed to show how the solution of any spherical triangle may be effected by the numerical or computing methods when three of its six parts are given. The graphical methods of solution will be considered in a later chapter. The following are the cases to be considered:—

Case I.—When no two of the given parts are consecutive.

(α) The three sides given.

(β) The three angles given.

Case II.—When two consecutive parts and a separated one are given.

(α) Two sides and the angle opposite one of them.

(β) Two angles and the side opposite one of them.

Case III.—When three consecutive parts are given.

(α) Two sides and the included angle.

(β) Two angles and the side between them.

We shall discuss these in order.

§ 35. **Case I(α). Three Sides given.**—Taking first the subcase (α), we have a, b, c given; and so at once by § 25—

$$\tan \tfrac{1}{2}\mathrm{A} = \sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}\right\}}$$

and similar formulæ for B and C.

In navigation (finding the hour-angle) this formula is frequently used in the form (§ 25)

$$\sin^2 \tfrac{1}{2}\mathrm{A} = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c};$$

use is then made of tables of the logarithm of $\sin^2 \frac{1}{2}\mathrm{A}$ (the *haversine* of A), which have been published with A as argument.

As a check, if only one angle A has been calculated (or as an alternative method of calculating A) we can use the equation

$$\cos A = \tan\theta \div \tan b,$$

where θ is an auxiliary angle defined by the equation

$$\tan(\tfrac{1}{2}c - \theta) = \tan\frac{a+b}{2}\tan\frac{a-b}{2} \div \tan\frac{c}{2};$$

or the equivalent equation (§ 17)

$$\cos(c-\theta)\cos b = \cos\theta\cos a, \quad \text{with} \quad \tan\theta = \cos A \tan b.$$

The auxiliary θ here introduced is easily seen to represent the intercept between A and the foot of the perpendicular from C on c. When all three angles are required, it is best to begin by calculating log m, where

$$m^2 = \sin(s-a)\sin(s-b)\sin(s-c) \div \sin s,$$

so that

$$\tan\tfrac{1}{2}A = \frac{m}{\sin(s-a)}, \text{ etc.}$$

One of Delambre's analogies (§ 29) may be employed in this case as a check.

Example 1.—Given $a = 70°\ 14'\ 20''{\cdot}12$, $b = 49°\ 24'\ 10''{\cdot}23$, $c = 38°\ 46'\ 10''{\cdot}34$, to find the angle A (less than 180°).

		$s =$	$79°\ 12'\ 20''{\cdot}345$
$a =$	$70°\ 14'\ 20''{\cdot}12$	$s-a =$	$8°\ 58'\ 0''{\cdot}225$
$b =$	$49°\ 24'\ 10''{\cdot}23$	$s-b =$	$29°\ 48'\ 10''{\cdot}115$
$c =$	$38°\ 46'\ 10''{\cdot}34$	$s-c =$	$40°\ 26'\ 10''{\cdot}005$
$2s =$	$158°\ 24'\ 40''{\cdot}69$		$158°\ 24'\ 40''{\cdot}69$

$$
\begin{aligned}
-\text{L}\sin s &= 0{\cdot}0077533{,}4\\
-\text{L}\sin(s-a) &= 0{\cdot}8072628{,}2\\
\text{L}\sin(s-b) &= 9{\cdot}6963708{,}0\\
\text{L}\sin(s-c) &= 9{\cdot}8119767{,}9\\
&\quad 20{\cdot}3233637{,}5\\
\text{L}\tan\tfrac{1}{2}A &= 10{\cdot}1616818{,}75\\
\tfrac{1}{2}A &= 55°\ 25'\ 37''{\cdot}42\\
\therefore\quad A &= 110°\ 51'\ 14''{\cdot}84
\end{aligned}
$$

Check.

$$
\begin{aligned}
\text{L}\tan b &= 10{\cdot}0670102{,}5\\
\text{L}\cos A &= 9{\cdot}5514376{,}1n\\
\text{L}\tan\theta &= 9{\cdot}6184478{,}6n\\
\theta &= -22°\ 33'\ 25''{\cdot}66\\
c-\theta &= 61°\ 19'\ 36''{\cdot}00
\end{aligned}
$$

$\text{L}\cos(c-\theta) =$	$9{\cdot}6810740{,}3$	$\text{L}\cos\theta =$	$9{\cdot}9654357{,}2$
$\text{L}\cos b =$	$9{\cdot}8134052{,}4$	$\text{L}\cos a =$	$9{\cdot}5290434{,}7$
	$9{\cdot}4944792{,}7$		$9{\cdot}4944791{,}9$

Example 2.—Find the angle B in the above triangle. (*Ans.*: 48° 56′ 5″·07.)

Example 3.—Given $a=70°\ 14'\ 20''\cdot12$, $b=49°\ 24'\ 10''\cdot23$, $c=321°\ 13'\ 49''\cdot66$, find all the angles if A is less than 180°.

$$\begin{array}{ll} & s=\ \ 220°\ 26'\ 10''\cdot005 \\ a=\ 70°\ 14'\ 20''\cdot12 & s-a=\ \ 150°\ 11'\ 49''\cdot885 \\ b=\ 49°\ 24'\ 10''\cdot23 & s-b=\ \ 171°\ \ 1'\ 59''\cdot775 \\ c=321°\ 13'\ 49''\cdot66 & s-c=-100°\ 47'\ 39''\cdot655 \\ 2s=440°\ 52'\ 20''\cdot01 & \qquad 440°\ 52'\ 20''\cdot01 \end{array}$$

$$\begin{array}{ll} \text{L}\sin(s-a)=9\cdot6963\ 708{,}2 & \\ \text{L}\sin(s-b)=9\cdot1927\ 372{,}0 & \\ \text{L}\sin(s-c)=9\cdot9922\ 466{,}4n & \\ -\text{L}\sin s\ \ =0\cdot1880\ 231{,}9n & \\ \qquad\qquad 9\cdot0693\ 778{,}5 & \\ \log m\ \ =9\cdot5346\ 889{,}25 & \\ -\text{L}\sin(s-a)=0\cdot3036\ 291{,}8 & \text{L}\tan\tfrac{1}{2}\text{A}=\ 9\cdot8383\ 181{,}05 \\ -\text{L}\sin(s-b)=0\cdot8072\ 628{,}0 & \text{L}\tan\tfrac{1}{2}\text{B}=10\cdot3419\ 517{,}25 \\ -\text{L}\sin(s-c)=0\cdot0077\ 533{,}6n & \text{L}\tan\tfrac{1}{2}\text{C}=\ 9\cdot5424\ 422{,}85n \\ \therefore\ \tfrac{1}{2}\text{A}=\ \ 34°\ 34'\ 22''\cdot570 & \text{A}=\ \ \ 69°\ \ 8'\ 45''\cdot14 \\ \tfrac{1}{2}\text{B}=\ \ 65°\ 31'\ 57''\cdot464 & \text{B}=\ 131°\ \ 3'\ 54''\cdot93 \\ \tfrac{1}{2}\text{C}=-19°\ 13'\ 23''\cdot764 & \text{C}=-\ 38°\ 26'\ 47''\cdot53 \\ & \quad =\ 321°\ 33'\ 12''\cdot47 \end{array}$$

Check.—Let us apply (§ 29)

$$\sin\tfrac{1}{2}(\text{A}+\text{B})\cos\tfrac{1}{2}c=\pm\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}\text{C}.$$

$$\begin{array}{ll} \tfrac{1}{2}(\text{A}+\text{B})=100°\ \ 6'\ 20''\cdot03 & \text{L}\sin\tfrac{1}{2}(\text{A}+\text{B})=9\cdot9932\ 096{,}9 \\ \tfrac{1}{2}c\ \ =160°\ 36'\ 54''\cdot83 & \text{L}\cos\tfrac{1}{2}c\ \ =9\cdot9746\ 548{,}7n \\ & \qquad 9\cdot9678\ 645{,}6n \\ a-b=20°\ 50'\ \ 9''\cdot89 & \\ \tfrac{1}{2}(a-b)=10°\ 25'\ 49''\cdot945 & \text{L}\cos\tfrac{1}{2}(a-b)=9\cdot9927\ 807{,}2 \\ & \text{L}\cos\tfrac{1}{2}\text{C}\ \ =9\cdot9750\ 837{,}1 \\ & \qquad 9\cdot9678\ 644{,}3 \end{array}$$

This shows good agreement: we notice that for this triangle the negative sign is to be taken in the Delambre group (§ 29).

Example 4.—Given $a=37°\ 42'\ 42''\cdot0$, b and c each equal to $29°\ 54'\ 30''\cdot2$, find A. (*Ans.*: 80° 48′ 28″·1.)

Example 5.—What does the triangle become when $a=b=c=120°$?

§ **36. Case I(β). Three Angles given.**—The formulæ when the given parts are the three angles are precisely similar to those of the preceding case: they are

$$\tan\tfrac{1}{2}a=\sqrt{\left\{\frac{-\cos\text{S}\cos(\text{S}-\text{A})}{\cos(\text{S}-\text{B})\cos(\text{S}-\text{C})}\right\}},\ \text{etc.,}$$

with the check (if only one side a is required)

$$\cos a=\tan\theta\div\tan\text{B},$$

where θ is defined by

$$\cot(\tfrac{1}{2}C+\theta)=\tan\frac{A+B}{2}\tan\frac{A-B}{2}\tan\frac{C}{2}.$$

If all three sides are required, we first calculate

$$M^2=-\cos S\div\{\cos(S-A)\cos(S-B)\cos(S-C)\},$$

when the sides are given by

$$\tan\tfrac{1}{2}a=M\cos(S-A),\text{ etc.},$$

and employ any one of Delambre's Analogies as a check.

Example.—Given $A=109°\ 45'\ 39''\cdot88$, $B=130°\ 35'\ 49''\cdot77$, $C=218°\ 46'\ 10''\cdot34$, find a and c. (*Ans.:* $a=110°\ 51'\ 14''\cdot86$, $c=218°\ 26'\ 47''\cdot53$.)

§ 37. **Case II(a). Two Sides and the Angle opposite one of them given.**—Case II. is distinguished as the *ambiguous case.* It has already been remarked that when three parts of a triangle are given there is always a certain ambiguity in determining the remaining parts; but in Cases I. and III. this does not correspond to any real physical ambiguity as regards the relative position of the three angular points of the triangle on the sphere, whereas in Case II. (which corresponds to the "ambiguous case" of Plane Trigonometry) there may be two physically different configurations of the angular points on the sphere, each of which satisfies the given conditions.

In Case II.(a), suppose that a, c, A, are given. Then C can be found from the equation

$$\sin C=\sin A\sin c\div\sin a.$$

The angle B can be found from the formula

$$\sin(B+\theta)=\sin\theta\tan c\cot a,$$

where

$$\tan\theta=\tan A\cos c$$

(it is easily seen that θ is the complement of the angle between c and the perpendicular from B on b); or, alternatively (if C has been already found), B can be found from one of Napier's Analogies—

$$\cot\tfrac{1}{2}B=\frac{\sin\frac{1}{2}(a+c)}{\sin\frac{1}{2}(a-c)}\tan\tfrac{1}{2}(A-C)=\frac{\cos\frac{1}{2}(a+c)}{\cos\frac{1}{2}(a-c)}\tan\tfrac{1}{2}(A+C).$$

The side b can be found either from the formula

$$\sin(b+\phi)=\cos a\sin\phi\div\cos c,$$

where

$$\cot \phi = \tan c \cos A$$

(it is easily seen that ϕ is the complement of the intercept between A and the foot of the perpendicular from B on b), or else (if C has been previously found) from one of Napier's Analogies—

$$\tan \tfrac{1}{2}b = \frac{\sin \frac{1}{2}(A+C)}{\sin \frac{1}{2}(A-C)} \tan \tfrac{1}{2}(a-c) = \frac{\cos \frac{1}{2}(A+C)}{\cos \frac{1}{2}(A-C)} \tan \tfrac{1}{2}(a+c).$$

As a check when C alone is required we can use equation (2) or (3) of § 14; if all the parts are required we can check by Delambre's Analogies.

Example 1.—Given $b = 49° \, 24' \, 10{\cdot}23$, $c = 321° \, 13' \, 49''{\cdot}66$, $B = 131° \, 3' \, 54''{\cdot}96$, to find C ($>270°$).

$$\begin{aligned} L \sin c &= 9{\cdot}7967\ 057{,}2n \\ L \sin B &= 9{\cdot}8773\ 493{,}2 \\ -L \sin b &= 0{\cdot}1195\ 845{,}6 \\ \hline L \sin C &= 9{\cdot}7936\ 396{,}0n \end{aligned} \qquad \begin{aligned} C &= -38°\ 26'\ 47''{\cdot}53 \\ &= 321°\ 33'\ 12''{\cdot}47 \end{aligned}$$

Check.—

$$\begin{aligned} B+C &= 92°\ 37'\ 7''{\cdot}43 \\ B-C &= 169°\ 30'\ 42''{\cdot}49 \\ \tfrac{1}{2}(B+C) &= 46°\ 18'\ 33''{\cdot}72 \\ \tfrac{1}{2}(B-C) &= 84°\ 45'\ 21''{\cdot}25 \end{aligned} \qquad \begin{aligned} b+c &= 370°\ 37'\ 59''{\cdot}89 \\ b-c &= -271°\ 49'\ 39''{\cdot}43 \\ \tfrac{1}{2}(b+c) &= 185°\ 18'\ 59''{\cdot}95 \\ \tfrac{1}{2}(b-c) &= -135°\ 54'\ 49''{\cdot}72 \end{aligned}$$

$\frac{1}{2}(B-C)$ is nearer 90° or 0° than $\frac{1}{2}(B+C)$, therefore write—

$$\tan \tfrac{1}{2}(B-C) = \tan \tfrac{1}{2}(B+C) \tan \tfrac{1}{2}(b-c) \cot \tfrac{1}{2}(b+c)$$

$$\begin{aligned} L \tan \tfrac{1}{2}(B+C) &= 0{\cdot}0198\ 565{,}3 \\ L \tan \tfrac{1}{2}(b-c) &= 9{\cdot}9861\ 445{,}1 \\ L \cot \tfrac{1}{2}(b+c) &= 1{\cdot}0312\ 353{,}2 \\ \hline L \tan \tfrac{1}{2}(B-C) &= 1{\cdot}0372\ 363{,}6 \end{aligned} \qquad \begin{aligned} \tfrac{1}{2}(B-C) &= 84°\ 45'\ 21''{\cdot}231 \\ B-C &= 169°\ 30'\ 42''{\cdot}46 \\ C &= 321°\ 33'\ 12''{\cdot}50 \end{aligned}$$

so that the value of C is increased by $0''{\cdot}03$.

Example 2.—If $a = 29° \, 54' \, 30''{\cdot}2$, $c = 36° \, 17' \, 10''{\cdot}0$, $A = 42° \, 41' \, 1''{\cdot}8$, show that the two possible values of b giving rise to distinct triangles are $47° \, 12' \, 44''{\cdot}0$ and $9° \, 30' \, 2''{\cdot}0$.

§ 38. **Case II(β). Two Angles and the Side opposite one of them given.**—This case so much resembles the last that it will not be necessary to do more than set down the formulæ.

Suppose A, C, a, are given. Then c is found from

$$\sin c = \sin a \sin C \div \sin A,$$

and b either from

$$\left\{ \begin{aligned} \tan \theta &= \tan a \cos C \\ \sin (b - \theta) &= \cot A \tan C \sin \theta \end{aligned} \right.$$

or from

$$\tan \tfrac{1}{2}b = \frac{\sin \frac{1}{2}(A+C)}{\sin \frac{1}{2}(A-C)} \tan \tfrac{1}{2}(a-c) = \frac{\cos \frac{1}{2}(A+C)}{\cos \frac{1}{2}(A-C)} \tan \tfrac{1}{2}(a+c);$$

while B is found either from

$$\begin{cases} \cot \phi = \cos a \tan C \\ \sin (B-\phi) = \cos A \sin \phi \div \cos C, \end{cases}$$

or from

$$\cot \tfrac{1}{2}B = \frac{\sin \frac{1}{2}(a+c)}{\sin \frac{1}{2}(a-c)} \tan \tfrac{1}{2}(A-C) = \frac{\cos \frac{1}{2}(a+c)}{\cos \frac{1}{2}(a-c)} \tan \tfrac{1}{2}(A+C).$$

Example 1.—Given $a=321°\ 13'\ 49''{\cdot}66$, $C=131°\ 3'\ 54''{\cdot}96$, $A=321°\ 33'\ 12''{\cdot}48$, find a value for b less than 90°.

$$\begin{array}{rl} L \tan a = & 9{\cdot}9047\ 943{,}4n \\ L \cos C = & \underline{9{\cdot}8175\ 113{,}3n} \\ L \tan \theta = & 9{\cdot}7223\ 056{,}7 \qquad \theta = 27°\ 48'\ 58''{\cdot}23 \\ -L \tan A = & 0{\cdot}1002\ 268{,}1n \\ L \sin \theta = & 9{\cdot}6689\ 785{,}6 \\ L \tan C = & \underline{10{\cdot}0598\ 379{,}5n} \\ L \sin (b-\theta) = & 9{\cdot}8290\ 433{,}2 \qquad \therefore\ b-\theta = 42°\ 25'\ 21''{\cdot}85 \end{array}$$

the supplement is excluded.

$\therefore\ b = 70°\ 14'\ 20''{\cdot}08.$

Check.—Apply the equation $\sin (c+A) \tan (\frac{1}{2}b-\theta) = \sin (c-A) \tan \frac{1}{2}b$.

$$\begin{array}{ll} C+A = 452°\ 37'\ 7''{\cdot}44 & b-2\theta = 14°\ 36'\ 23''{\cdot}62 \\ C-A = 190°\ 29'\ 17''{\cdot}52 & \tfrac{1}{2}(b-2\theta) = 7°\ 18'\ 11''{\cdot}81 \\ & \tfrac{1}{2}b = 35°\ 7'\ 10''{\cdot}04 \\ (\textit{Check}) & \theta = 27°\ 48'\ 58''{\cdot}23 \end{array}$$

$$\begin{array}{ll} L \sin (C+A) = 9{\cdot}9995\ 462{,}1 & L \tan \tfrac{1}{2}b = 9{\cdot}8471\ 524{,}5 \\ L \tan \tfrac{1}{2}(b-2\theta) = \underline{9{\cdot}1077\ 563{,}8} & L \sin (C-A) = \underline{9{\cdot}2601\ 502{,}0} \\ 9{\cdot}1073\ 026{,}9 & 9{\cdot}1073\ 026{,}5. \end{array}$$

Example 2.—Given $a=130°\ 13'\ 49''{\cdot}9$, $A=150°\ 5'\ 29''{\cdot}8$, $C=148°\ 17'\ 43''{\cdot}6$, show that one of the two possible values of B which give distinct triangles is $139°\ 23'\ 38''{\cdot}0$, and find the other.

§ 39. Case III(a). Two Sides and the Included Angle given.—Taking next the case when two sides and the included angle—say a, b, C—are given, we can proceed by introducing an auxiliary angle θ, equal to the intercept between C and the foot of the perpendicular from B on b; θ is to be found from the equation

$$\tan \theta = \tan a \cos C \quad . \quad . \quad . \quad . \quad (1)$$

and c and A are then given immediately by

$$\cos c = \cos a \cos (b-\theta) \div \cos \theta \quad . \quad . \quad . \quad (2)$$

$$\tan A = \sin \theta \tan C \div \sin (b-\theta) \quad . \quad . \quad . \quad (3)$$

while B is finally derived from

$$\sin B = \sin C \sin b \div \sin c = \sin A \sin b \div \sin a.$$

A second method (though not essentially distinct in theory) is to introduce in place of θ an auxiliary ϕ equal to the intercept between C and the foot of the perpendicular from A on a: ϕ is found from

$$\tan \phi = \tan b \cos C,$$

and c and B are then given immediately by

$$\cos c = \cos b \cos (a-\phi) \div \cos \phi$$
$$\tan B = \sin \phi \tan C \div \sin (a-\phi),$$

and A is derived from the sine formula.

A third method is to derive A and B first by Napier's Analogies—

$$\tan \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2}(a-b) \cot \tfrac{1}{2}C \div \cos \tfrac{1}{2}(a+b),$$
$$\tan \tfrac{1}{2}(A-B) = \sin \tfrac{1}{2}(a-b) \cot \tfrac{1}{2}C \div \sin \tfrac{1}{2}(a+b),$$

and then find c from the sine formula.

If only *one* of the unknown parts is required, we naturally choose that method which gives the part in question most directly.

Example 1 (*illustrating the first method*).—

Given $a=49° 24' 10''{\cdot}23$, $C=69° 8' 45''{\cdot}16$, $b=321° 13' 49''{\cdot}66$, to find $A<180°$.

$$\begin{aligned} L\tan a &= 10{\cdot}0670\ 102,5 \\ L\cos C &= 9{\cdot}5514\ 376,1 \\ \hline L\tan\theta &= 9{\cdot}6184\ 478,6 \qquad & \theta &= 22°\ 33'\ 25''{\cdot}66 \\ & & b-\theta &= 298°\ 40'\ 24''{\cdot}00 \\ -L\sin(b-\theta) &= 0{\cdot}0568\ 174,2n \\ L\sin\theta &= 9{\cdot}5838\ 836,2 \\ L\tan C &= 0{\cdot}4191\ 369,7 \\ \hline L\tan A &= 0{\cdot}0598\ 380,1n & A &= -\ 48°\ 56'\ 5''{\cdot}06 \\ & & &= 131°\ 3'\ 54''{\cdot}94. \end{aligned}$$

As a check we can use the equation $\sin(C+A)\tan(\tfrac{1}{2}b-\theta)=\sin(C-A)\tan\tfrac{1}{2}b$.

$$\begin{aligned} C+A &= 200°\ 12'\ 40''{\cdot}10 \qquad & b-2\theta &= 276°\ 6'\ 58''{\cdot}34 \\ C-A &= -61°\ 55'\ 9''{\cdot}78 & \tfrac{1}{2}(b-2\theta) &= 138°\ 3'\ 29''{\cdot}17 \\ & & \tfrac{1}{2}b &= 160°\ 36'\ 54''{\cdot}83 \\ & (\textit{Check}) & \theta &= 22°\ 33'\ 25''{\cdot}66 \end{aligned}$$

$$\begin{aligned} L\sin(C+A) &= 9{\cdot}5384\ 237,5n \qquad & L\tan\tfrac{1}{2}b &= 9{\cdot}5463\ 660,3n \\ L\tan\tfrac{1}{2}(b-2\theta) &= 9{\cdot}9535\ 516,9n & L\sin(C-A) &= 9{\cdot}9456\ 094,8n \\ \hline & 9{\cdot}4919\ 754,4 & & 9{\cdot}4919\ 755,1 \end{aligned}$$

Example 2.—Given $a=43° 30' 2''{\cdot}0$, $b=31° 42' 16''{\cdot}4$, $C=49° 46' 10''{\cdot}1$, find A.

(*Ans.*: $89° 40' 35''{\cdot}4$.)

Example 3 (*illustrating the third method*).—

Given $a=49° \, 24' \, 10''\cdot 23$, $b=321° \, 13' \, 49''\cdot 66$, $C=69° \, 8' \, 45''\cdot 16$, find A and B if c be less than 180°.

$$a+b= 370° \, 37' \, 59''\cdot 89 \qquad \tfrac{1}{2}(a+b)= 185° \, 18' \, 59''\cdot 945$$
$$a-b=-271° \, 49' \, 39''\cdot 43 \qquad \tfrac{1}{2}(a-b)=-135° \, 54' \, 49''\cdot 715$$
$$\tfrac{1}{2}C= 34° \, 34' \, 22''\cdot 58 \qquad a= 49° \, 24' \, 10''\cdot 23$$
$$b= 321° \, 13' \, 49''\cdot 66$$

$L\cos\frac{1}{2}(a-b) = 9\cdot 8563\ 022{,}4n$	$L\sin\frac{1}{2}(a-b) = 9\cdot 8424\ 467{,}9n$
$-L\cos\frac{1}{2}(a+b) = 0\cdot 0018\ 724{,}7n$	$-L\sin\frac{1}{2}(a+b) = 1\cdot 0331\ 079{,}7n$
$-L\tan\frac{1}{2}C = 0\cdot 1616\ 818{,}4$	$-L\tan\frac{1}{2}C = 0\cdot 1616\ 818{,}4$
$L\tan\frac{1}{2}(A+B)=10\cdot 0198\ 565{,}5$	$L\tan\frac{1}{2}(A-B)=11\cdot 0372\ 366{,}0$
$\frac{1}{2}(A+B)=46° \, 18' \, 33''\cdot 72$	$A= 131° \, 3' \, 54''\cdot 96$
$\frac{1}{2}(A-B)=84° \, 45' \, 21''\cdot 24$	$B=- \, 38° \, 26' \, 47''\cdot 52$

The equation $\sin c \sin B = \sin C \sin b$ tells us, since $\sin c$ is positive, that $\sin B$ is negative, so that we do not need to add 180° to these results. We therefore write—

$$B=321° \, 33' \, 12''\cdot 48.$$

Check.—Applying equation (2).

$L\tan\frac{1}{2}(A+B)=10\cdot 0198\ 565{,}5$	$L\tan\frac{1}{2}(A-B)=11\cdot 0372\ 366{,}0$
$L\tan\frac{1}{2}(a-b) = 9\cdot 9861\ 445{,}5$	$L\tan\frac{1}{2}(a+b) = 8\cdot 9687\ 645{,}0$
$0\cdot 0060\ 011{,}0$	$0\cdot 0060\ 011{,}0$

§ 40. **Case III(β). Two Angles and the Side between them given.**—This case is precisely similar to the last. If c, A, B, are given, we can define an auxiliary θ by the equation

$$\tan\theta = \cos c \tan A,$$

and then calculate a and C from

$$\tan a = \tan c \sin\theta \div \sin(B+\theta)$$
$$\cos C = \cos A \cos(B+\theta) \div \cos\theta,$$

and b from the sine formula. Or we can define an auxiliary ϕ by the equation

$$\tan\phi = \cos c \tan B,$$

and then calculate b and C from

$$\tan b = \tan c \sin\phi \div \sin(A+\phi)$$
$$\cos C = \cos B \cos(A+\phi) \div \cos\phi$$

and a from the sine formula. Or, lastly, we can derive a and b first from

$$\tan\tfrac{1}{2}(a+b) = \cos\tfrac{1}{2}(A-B) \tan\tfrac{1}{2}c \div \cos\tfrac{1}{2}(A+B)$$
$$\tan\tfrac{1}{2}(a-b) = \sin\tfrac{1}{2}(A-B) \tan\tfrac{1}{2}c \div \sin\tfrac{1}{2}(A+B),$$

and then find C from the sine formula.

Example.—Given $c=36° \, 17' \, 10''\cdot 0$, $A=106° \, 44' \, 44''\cdot 7$, $B=42° \, 41' \, 1''\cdot 8$, find a.

(*Ans.:* $56° \, 42' \, 46''\cdot 0$.)

§ 41. **Solution by Right-angled Triangles.**—A triangle which is not right-angled can always be solved by dividing it into two right-angled triangles. For suppose, first, that among the three given parts there are two which are consecutive. We can without loss of generality take these to be a and C. Draw a perpendicular BD to the side a (fig. 7). Then the knowledge of a and C enables us to solve completely the right-angled triangle BCD; and then, knowing BD and the third given part, which must furnish some other datum relative to the triangle ABD, we can solve the latter right-angled triangle, and so complete the solution of the original triangle. If of the three given parts no two are consecutive, the three given parts must be either the three sides or the three angles. Supposing, first, that the three sides a, b, c are given, we have from fig. 7—

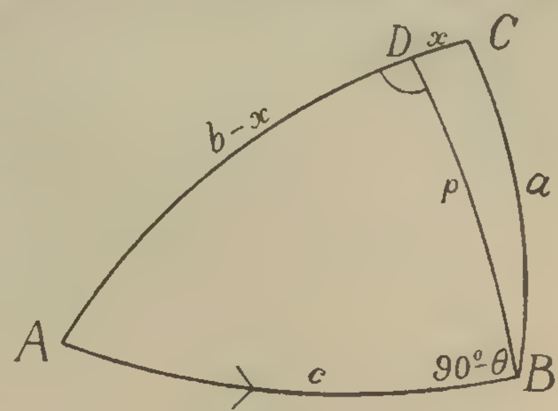

FIG. 7.

$$\frac{\cos(b-x)}{\cos c}=\frac{1}{\cos p}=\frac{\cos x}{\cos a}.$$

Hence

$$\frac{\cos(b-x)-\cos x}{\cos(b-x)+\cos x}=\frac{\cos c-\cos a}{\cos c+\cos a}$$

or

$$\tan(x-\tfrac{1}{2}b)=\tan\tfrac{1}{2}(a-c)\tan\tfrac{1}{2}(a+c)\cot\tfrac{1}{2}b.$$

This equation determines x, and therefore $(b-x)$: the values of A and C are then given by

$$\cos A=\tan(b-x)\cot c$$
$$\cos C=\tan x\cot a,$$

and B can be obtained from the sine formula.

Similarly when the three angles are given, we can determine the angle CBD in terms of them, and hence, by solving the right-angled triangles BCD and BAD, obtain a, b, and c.

Example 1.—If $a=97°\ 26'\ 29''\cdot 0$, $C=95°\ 38'\ 4''\cdot 5$, $A=82°\ 33'\ 31''\cdot 0$, find b. ($b=115°\ 36'\ 44''\cdot 8$.)

Example 2.—If a and b are as above, and $c=99°\ 40'\ 48''\cdot 5$, find B. ($B=114°\ 26'\ 49''\cdot 8$.)

CHAPTER V

SPECIAL APPLICATIONS

§ 42. **Introductory.**—The most frequent use of spherical trigonometry is in problems connected with astronomy and navigation. In the former a high degree of accuracy is generally demanded; but in the latter less accuracy is required, and we may sometimes make use of one or other of the graphical processes to be described in the next chapter. We now proceed to mention a few frequently recurring types of problems; the reader is referred to technical works for complete detail.

§ 43. **Great Circle Sailing.**—It is easily seen that the shortest distance between two points on a sphere is along the great circle joining them; so that, other things being equal, a ship should endeavour to follow this circle from port to port, and it becomes a practical problem to determine the direction to pursue at any part of the course.

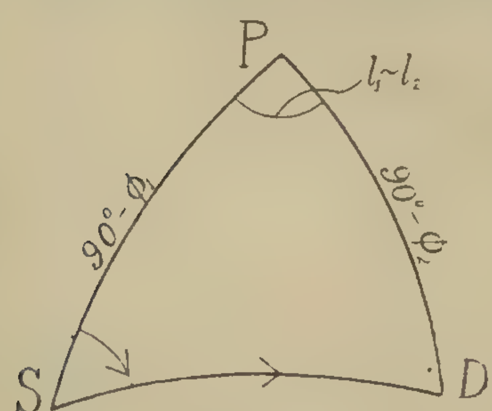

FIG. 8.

Let the ship's present position S be latitude ϕ_1 and longitude l_1, and let the corresponding values for the destination D be (ϕ_2, l_2). Then in the triangle PSD (fig. 8), where P is the north pole, the sides PS and PD are respectively $90°-\phi_1$ and $90°-\phi_2$, and the contained angle is the difference of longitude $l_1 \sim l_2$. The angle at S gives the direction measured from the north, and, proceeding as in § 39 with the four consecutive parts $90°-\phi_2$, $l_1 \sim l_2$, $90°-\phi_1$, S, we have by equation (3)

$$\tan S = \sin\theta \tan(l_1 \sim l_2) \div \sin(90° - \phi_1 - \theta)$$

$$i.e. \quad \tan S = \frac{\sin\theta \tan(l_1 \sim l_2)}{\cos(\phi_1 + \theta)},$$

where by (1)

$$\tan\theta=\tan(90^\circ-\phi_2)\cos(l_1\sim l_2)=\cos(l_1\sim l_2)/\tan\phi_2.$$

The distance SD, if required, is given by (2)

$$\cos SD\cos\theta=\cos(90^\circ-\phi_1-\theta)\cos(90^\circ-\phi_2)$$
$$=\sin(\phi_1+\theta)\sin\phi_2.$$

The latitudes used above have been measured from the equator northwards. If south latitude is given, we must of course change the sign of the corresponding ϕ in the above expressions.

Example.—Find the initial course to be steered (and the distance) by a ship sailing from Strait of Belle Isle (say 52° N. and 55° W.) for Land's End (50° N. and 6° W.).

$$\begin{array}{lll}
\log\cos(l_1\sim l_2)=\log\cos 49^\circ=9{\cdot}8169 & & \\
\log\tan\phi_2 \quad =\log\tan 50^\circ=\ \underline{{\cdot}0762} & & \\
\log\tan\theta \quad =9{\cdot}7407 & & \theta\doteq 28^\circ\ 2' \\
\log\sin\theta=9{\cdot}6721 & & \phi_1+\theta=80^\circ\ 2' \\
\log\tan 49^\circ=\underline{0{\cdot}0608} & & \log\cos(\phi_1+\theta)=9{\cdot}2382 \\
9{\cdot}7329 & & \\
\underline{1{\cdot}2382} & & \\
\log\tan S=\ {\cdot}4947 & S=72^\circ\ 15' & \\
\log\sin\phi_2=9{\cdot}8843 & & \\
\log\sin(\phi_1+\theta)=\underline{9{\cdot}9934} & & \\
9{\cdot}8777 & & \\
\log\cos\theta=9{\cdot}9458 & & \\
\log\cos SD=9{\cdot}93\overline{19} & & SD=31^\circ\ 16'=1876'=1876\text{ miles}
\end{array}$$

(*i.e.* the course to be steered initially is $17\frac{3}{4}^\circ$ North of East.)

(1′ = 1 nautical mile).

1. Find the point at which the ship, if it follows the great circle, will be sailing due east. (54° N.: 33° 54′ W.)
2. Find the distance from the same point of departure to a point due east (*i.e.* in the same latitude) and 12° W. (1564 miles.)
3. Find the distance in question 2 if the ship sailed due east, keeping a constant course. (1588 miles.)

§ 44. **Finding the Longitude at Sea.**—Let a ship's position at time t_1, as indicated by a chronometer giving Greenwich time, be latitude ϕ measured positively northwards from the equator, and longitude l measured west from the Greenwich meridian. Let the sun's altitude above the horizon be a at the same time. Then in the triangle ZPS (fig. 9), where P is the north pole, Z is the zenith, and S the position of the sun, we have $SZ=(90^\circ-a)$, and $SP=(90^\circ-\delta)$, where δ is the declination north of the celestial

equator and ZP is the angle between the lines joining the centre of the earth to the Pole, and to the place in question, *i.e.* $ZP=(90°-\phi)$. The altitude a is read by means of a sextant, δ is given by the *Nautical Almanac* for small intervals of time throughout the year, and ϕ is determined as follows. At midday the sailor observes the sun's greatest altitude a', whence, since the triangle has then closed up so that the angle at P is zero,

$$S'P=S'Z+ZP, \quad i.e. \quad 90°-\delta'=90°-a'+90°-\phi',$$

which gives ϕ'. Now, knowing the ship's component velocity northwards, we easily find the change of latitude between the times of observation and thus determine ϕ. Three sides of the triangle are now known, and the angle h at P (or the *hour-angle*, as it is called) can be calculated.

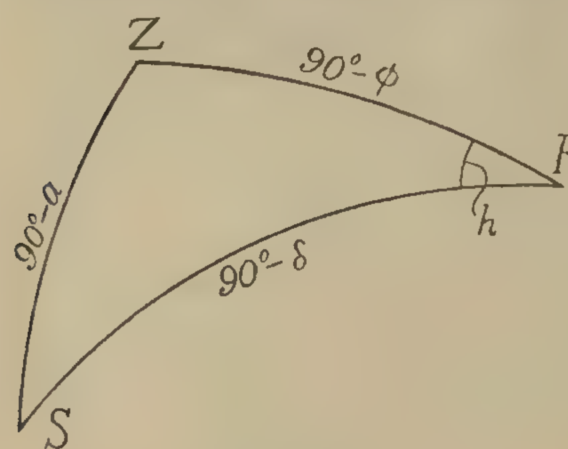

FIG. 9.

As mentioned in § 35, the sine of the half-angle is usually used, and, inserting the expressions for the sides in formula (9), we easily obtain

$$\sin^2 \tfrac{1}{2}h=\frac{\sin \frac{1}{2}(90°+a+\delta+\phi)\sin \frac{1}{2}(90°+a-\delta-\phi)}{\cos \delta \cos \phi},$$

which determines h.

Multiplying t_1 by 15 to convert it into degrees of longitude, we have the distance in longitude that Greenwich was eastward of the sun's meridian (*i.e.* of the meridian PS) at the time of observation. Adding h (or subtracting if the observation was taken *after* local noon) we have the ship's longitude west of Greenwich as required, *i.e.* $l=15t_1+h$.

The time t_1 should be chosen when the sun's altitude is changing rapidly in order to make as small as possible the error in determining the time t_1 at which the sun reached the altitude a.

Example 1.—In latitude 43°, the sun's declination being 16° N., the altitude of the sun's centre is (all reductions having been made) 23°. Find the hour-angle. (*Ans.:* $4^h\ 52^m\ 57^s$.)

Example 2.—In latitude 36°, the sun's declination being 13° N., the altitude of the sun's centre is 10°. Find the hour-angle. (*Ans.*: $5^h\ 47^m\ 57^s$.)

Example 3.—In latitude 45° 25′ 30″, the sun's declination being 23° 28′ 30″ N., the altitude is 2° 25′ 30″. Find the hour-angle. (*Ans.*: $7^h\ 28^m\ 14^s$.)

§ **45. Reducing an Angle to the Horizon.**—In survey work we are sometimes given the altitudes a_1 and a_2 of two objects A and B as seen from a station O, together with the angle they subtend at O, while we require to insert their positions among others on a chart which can only give their projections. It is therefore necessary to calculate the angle which the line joining the projections of O and A makes with the line joining the projections of O and B.

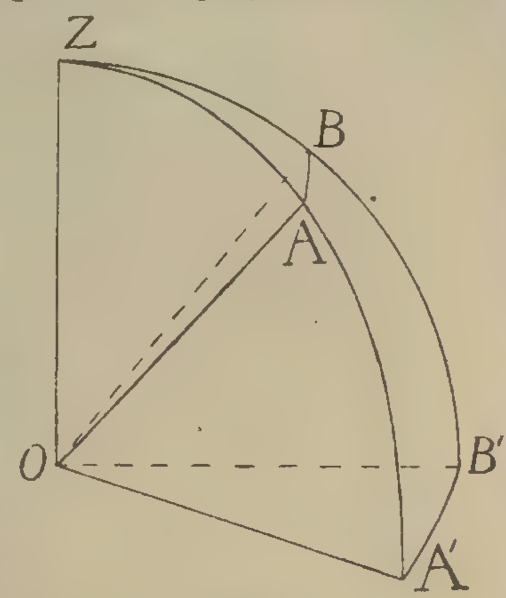

FIG. 10.

Through the points A and B draw a sphere (fig. 10) having O as centre, and join the zenith Z to the points A and B by two great circles meeting the horizontal plane through O in A', B'. Then, since the vertical lines through A and B meet the lines OA' and OB' respectively, the required angle is $A'OB'$, *i.e.* the angle Z of the triangle ZAB. The three sides of this triangle are given for $ZA = 90° - a_1$, $ZB = 90° - a_2$, and AB is the observed angle subtended by A and B. We can therefore obtain the required value by the formula (10) for $\tan \frac{1}{2} Z$. In this case, however, the quantities a_1 and a_2 are usually small, so that we can use an approximation. Let the angle AOB be X, and let the angle at Z be $X+x$ so that x is small, then by the cosine formula we have—

$$\cos(X+x) = \frac{\cos X - \cos AZ \cos BZ}{\sin AZ \sin BZ}$$

$$i.e. \qquad \cos X - x \sin X = \frac{\cos X - \sin a_1 \sin a_2}{\cos a_1 \cos a_2}$$

$$= \frac{\cos X - a_1 a_2}{1 - \frac{1}{2}(a_1{}^2 + a_2{}^2)}$$

(if we stop at squares of a_1 and a_2).

This can be written

$$x \sin X = a_1 a_2 - \tfrac{1}{2}(a_1{}^2 + a_2{}^2) \cos X,$$

or

$$x = \tfrac{1}{4}(a_1 + a_2)^2 \tan \tfrac{1}{2}X - \tfrac{1}{4}(a_1 - a_2)^2 \cot \tfrac{1}{2}X.$$

§ 46. **Conversion of Star Coordinates.**—Astronomers denote the position of a celestial body by one of two systems of coordinates, each being exactly similar to the system described in § 6. Both systems have the earth's centre O (fig. 11) as the centre of the sphere of reference, but while one has for pole P, the direction of the earth's axis, and a corresponding equator

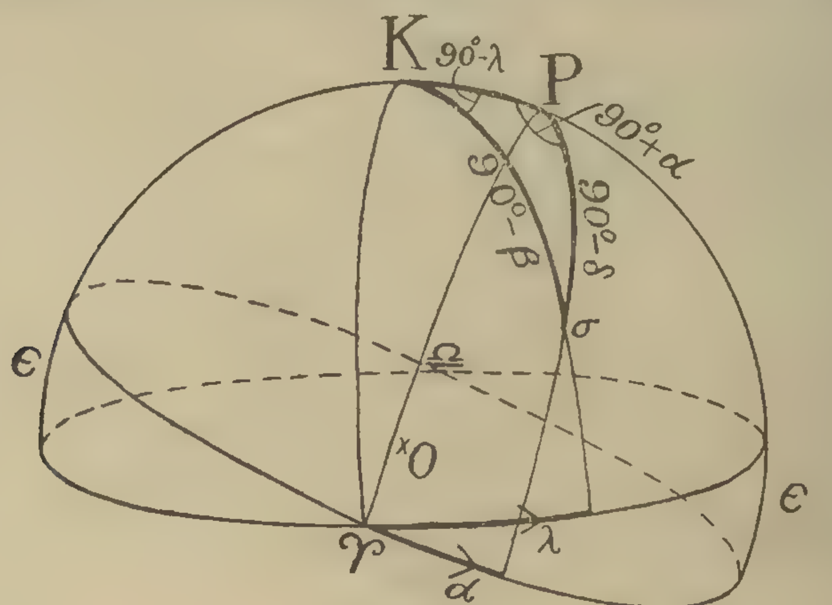

FIG. 11.

♈a♎, the other has its pole at a well-defined point K about 23° distant from P with a corresponding equator ♈λ♎ called the *ecliptic*. If we draw meridians $P\sigma$ and $K\sigma$ through the star's position σ, then plainly the star's position can be given either in terms of a (*right ascension*) and δ (*declination*), defined as in the figure, or it can be given in terms of λ (*longitude*) and β (*latitude*). Now, it is a frequently recurring problem to change from the one set of coordinates to the other, and this can be done directly by means of the triangle $PK\sigma$, for it involves all the coordinates a, β, δ, λ, and three of its parts are always given. Further, by the nature of the problem neither δ nor β can exceed $\pm 90°$, so that the solution of the triangle is unique (§ 11).

Problem I.—*Given the right ascension a of a star and its declination δ, to find its longitude λ and its latitude β.*

We have two sides and the included angle of the triangle $PK\sigma$ given, and require the next two in order. Therefore, by § 39, writing

$$\tan\theta = \tan(90° - \delta)\cos(90° + \alpha) = -\sin\alpha\cot\delta,$$

we have, by equation (2)—

$$\cos(90° - \beta) = \cos(\epsilon - \theta)\cos(90° - \delta) \div \cos\theta$$

i.e. $$\sin\beta = \cos(\epsilon - \theta)\sin\delta/\cos\theta \quad . \quad . \quad . \quad (1)$$

which determines β uniquely, since it must be in the first or fourth quadrant.

To determine λ we use equation (3) and the Sine Formula, one for the calculation and the other as a quadrant check.

$\therefore$ $$\sin(\epsilon - \theta)\tan(90° - \lambda) = \sin\theta\tan(90° + \alpha)$$

and $$\sin(90° - \beta)\sin(90° - \lambda) = \sin(90° + \alpha)\sin(90° - \delta)$$

or $$\tan\lambda = \tan\alpha\sin(\theta - \epsilon)/\sin\theta \quad . \quad . \quad . \quad (2)$$

$$\cos\lambda = \cos\alpha\cos\delta/\cos\beta \quad . \quad . \quad . \quad . \quad (3)$$

By calculating λ from both (2) and (3) we have a partial check on our working.

Using (1) and (3) we obtain a value for $\tan\beta$—the tangent giving more accurate values than the sine or cosine—and arrange the formulæ thus

$$\left.\begin{aligned} \tan\theta &= -\sin\alpha/\tan\delta \\ \tan\lambda &= \frac{\sin(\theta - \epsilon)\tan\alpha}{\sin\theta} \\ \tan\beta &= \frac{\sin\lambda}{\tan(\epsilon - \theta)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

determining the quadrant of λ by (3)

$$\cos\beta\cos\lambda = \cos\alpha\cos\delta.$$

If either λ or a be very small, the second or third of equations (4) may turn out to be the ratio of small quantities. To avoid this computers introduce a second auxiliary m, so that the equations become

$$m\sin\theta = \sin(90° - \delta)\cos(90° + \alpha) = \sin(-\alpha)\cos\delta \quad . \quad (5)$$

$$m\cos\theta = \cos(90° - \delta) \qquad = \sin\delta \quad . \quad . \quad . \quad (6)$$

$$\sin\beta = m\cos(\epsilon - \theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\sin\lambda\cos\beta = m\sin(\epsilon - \theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\cos\lambda\cos\beta = \cos\alpha\cos\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Tan θ is determined by subtracting the logs of (5) and (6); and then, looking up $\log \sin \theta$ or $\log \cos \theta$ (whichever is numerically the greater), we have $\log m$ from (5) and (6).*

Tan λ is obtained by subtracting the logs of (8) and (9), and from $\log \sin \lambda$ or $\log \cos \lambda$ (whichever is the greater) and (8) or (9) we determine $\cos \beta$. Sin β is given by (7) and, subtracting from it the value of $\cos \beta$, we determine β by its tangent. Entering the tables with whichever is the smaller, $\sin \beta$ or $\cos \beta$, we determine a second value for β, the possession of the two determinations serving as a partial check on the working.

Check Equations.—Applying the two Napier analogies to the five parts $90° - \delta$, $90° + \alpha$, ϵ, $90° - \lambda$, $90° - \beta$, we easily obtain from (18) and (19)—

$$\sin \tfrac{1}{2}(\lambda - \alpha) = \tan \tfrac{1}{2}\epsilon \cos \tfrac{1}{2}(\lambda + \alpha) \tan \tfrac{1}{2}(\quad + \beta)$$
$$\tan \tfrac{1}{2}(\delta - \beta) = \tan \tfrac{1}{2}\epsilon \sin \tfrac{1}{2}(\lambda + \alpha) \sec \tfrac{1}{2}(\lambda - \alpha).$$

Since the differences $\frac{1}{2}(\lambda - \alpha)$ and $\frac{1}{2}(\delta - \beta)$ are usually small, the right-hand side of these equations is less affected by errors in λ and β than the left-hand sides; we therefore obtain a closer approximation by substituting the values already obtained in the right-hand sides and then recalculating $\lambda - \alpha$ and $\delta - \beta$, accepting the resulting values as final. They will be better values than the preceding ones on account of the avoidance of the auxiliary angle.

As an example of the actual calculation let us convert the coordinates of γ Pegasi (1911).

$$\alpha = 2° \, 10' \, 5''{\cdot}10, \quad \delta = 14° \, 41' \, 24''{\cdot}17, \quad \epsilon = 23° \, 27' \, 3''{\cdot}11$$

$\cos\lambda \cos\beta = 9{\cdot}9852\ 555$	$\sin\beta = 9{\cdot}3386\ 170$
$\cos\alpha = 9{\cdot}9996\ 890$	$\cos(\epsilon - \theta) = 9{\cdot}9300\ 095$
$\cos\delta = 9{\cdot}9855\ 665$	$m = 9{\cdot}4086\ 075$
$\sin(-\alpha) = 8{\cdot}5778\ 497n$	$\sin(\epsilon - \theta) = 9{\cdot}7200\ 863$
$m \sin\theta = 8{\cdot}5634\ 162n$	$\sin\lambda \cos\beta = 9{\cdot}1286\ 938$
$\sin\delta = m\cos\theta = 9{\cdot}4041\ 320$	$\cos\lambda \cos\beta = 9{\cdot}9852\ 555$
$\tan\theta = 9{\cdot}1592\ 842n$	$\tan\lambda = 9{\cdot}1434\ 383$
$\theta = -\ 8° \, 12' \, 41''{\cdot}17$	$\lambda = 7° \, 55' \, 15''{\cdot}71$
$\epsilon - \theta = 31° \, 39' \, 44''{\cdot}28$	sine or cos $\lambda = 9{\cdot}9958\ 365$
sine or cos $\theta = 9{\cdot}9955\ 245$	$\cos\beta = 9{\cdot}9894\ 190$
	$\tan\beta = 9{\cdot}3491\ 980$
	$\beta = 12° \, 35' \, 46''{\cdot}76$
Check (by $\sin\beta$)	$\beta = 12° \, 35' \, 46''{\cdot}75$

* See § 3.

Application of Check equations.—

$\delta = 14° \ 41' \ 24''\cdot17$	$\lambda = 7° \ 55' \ 15''\cdot71$
$\beta = 12° \ 35' \ 46''\cdot76$	$\alpha = 2° \ 10' \ 5''\cdot10$
$\delta+\beta = 27° \ 17' \ 10''\cdot93$	$\lambda+\alpha = 10° \ 5' \ 20''\cdot81$
$\delta-\beta = 2° \ 5' \ 37''\cdot41$	$\lambda-\alpha = 5° \ 45' \ 10''\cdot61$
$\frac{1}{2}(\delta+\beta) = 13° \ 38' \ 35''\cdot47$	$\frac{1}{2}(\lambda+\alpha) = 5° \ 2' \ 40''\cdot42$
$\frac{1}{2}(\delta-\beta) = 1° \ 2' \ 48''\cdot71$	$\frac{1}{2}(\lambda-\alpha) = 2° \ 52' \ 35''\cdot31$

$$\tfrac{1}{2}\epsilon = 11° \ 43' \ 31''\cdot56$$

$\sin\frac{1}{2}(\lambda+\alpha) = 8\cdot9441\ 394{,}9$	$\cos\frac{1}{2}(\lambda+\alpha) = 9\cdot9983\ 145{,}5$
$-\cos\frac{1}{2}(\lambda-\alpha) = 0\cdot0005\ 475{,}3$	$\tan\frac{1}{2}(\delta+\beta) = 9\cdot3851\ 116{,}7$
$\tan\frac{1}{2}\epsilon = 9\cdot3171\ 289{,}9$	$\tan\frac{1}{2}\epsilon = 9\cdot3171\ 289{,}9$
$\tan\frac{1}{2}(\delta-\beta) = 8\cdot2618\ 160{,}1$	$\sin\frac{1}{2}(\lambda-\alpha) = 8\cdot7005\ 552{,}1$
$\frac{1}{2}(\delta-\beta) = 1° \ 2' \ 48''\cdot711$	$\frac{1}{2}(\lambda-\alpha) = 2° \ 52' \ 35''\cdot301$
$\delta-\beta = 2° \ 5' \ 37''\cdot42$	$\lambda-\alpha = 5° \ 45' \ 10''\cdot60$
$\beta = 12° \ 35' \ 46''\cdot75$	$\lambda = 7° \ 55' \ 15''\cdot70$

In the check equations the utmost accuracy of the tables is made use of, as we depend on this stage of the calculation for the final values of λ and β.

§ **47. Problem 2.**—*Given the longitude λ of a star and its latitude β, to find its right ascension α and its declination δ.*

Referring again to the triangle $PK\sigma$ (fig. 11), we see that again we are given two sides and the included angle. We therefore write

$$\tan\theta = \tan(90° - \beta)\cos(90° - \lambda) = \sin\lambda\cot\beta,$$

and proceed as before.

We may, however, evidently interchange in the triangle β and δ, and also $-\lambda$ and α. In this way equations (4) become

$$\left.\begin{aligned} \tan\theta &= \sin\lambda/\tan\beta \\ \tan\alpha &= \frac{\sin(\theta-\epsilon)\tan\lambda}{\sin\theta} \\ \tan\delta &= \frac{\sin\alpha}{\tan(\theta-\epsilon)} \end{aligned}\right\}$$

To avoid difficulties when α and λ are small, recourse is usually had to the extended form involving m: the equations become

$$\begin{aligned} m\sin\theta &= \sin\lambda\cos\beta \\ m\cos\theta &= \sin\beta \\ \sin\delta &= m\cos(\theta-\epsilon) \\ \sin\alpha\cos\delta &= m\sin(\theta-\epsilon) \\ \cos\alpha\cos\delta &= \cos\lambda\cos\beta, \end{aligned}$$

where the same procedure in solving is adopted as in last section.

Example.—Given * $\lambda=283°\ 54'\ 51''\cdot36$, $\beta=61°\ 44'\ 16''\cdot79$, $\epsilon=23°\ 27'\ 8''\cdot26$, to calculate α and δ.

$\cos\alpha\cos\delta=9\cdot0563\ 842$	$\sin\delta=9\cdot7959\ 585$
$\cos\lambda=9\cdot3810\ 604$	$\cos(\theta-\epsilon)=9\cdot7987\ 922$
$\cos\beta=9\cdot6753\ 238$	$m=9\cdot9971\ 663$
$\sin\lambda=9\cdot9870\ 656n$	$\sin(\theta-\epsilon)=9\cdot8905\ 547n$
$m\sin\theta=9\cdot6623\ 894n$	$\sin\alpha\cos\delta=9\cdot8877\ 210n$
$\sin\beta=m\cos\theta=9\cdot9448\ 732$	$\cos\alpha\cos\delta=9\cdot0563\ 842$
$\tan\theta=9\cdot7175\ 162n$	$\tan\alpha=0\cdot8313\ 368$
$\theta=-27°\ 33'\ 22''\cdot35$	$\alpha=-\ 81°\ 36'\ 42''\cdot57$
$\theta-\epsilon=-51°\ 0'\ 30''\cdot61$	$=\ 278°\ 23'\ 17''\cdot43$
sine or cos $\theta=9\cdot9477\ 069$	sine or cos $\alpha=9\cdot9753\ 291n$
	$\cos\delta=9\cdot8923\ 919$
	$\tan\delta=9\cdot9035\ 666$
	$\delta=38°\ 41'\ 25''\cdot74$
Check (by $\sin\delta$)	$\delta=38°\ 41'\ 25''\cdot72$

On applying the same check as in the preceding example, we shall decide finally the values of α and δ to accept. The procedure is identical with that of the last section.

Examples.—Using the two auxiliaries $(m,\ \theta)$ defined by $m\sin\theta=\sin a\cos C$, $m\cos\theta=\cos a$, find A and c in the following cases—

(1) $a=64°\ 23'\ 15''\cdot2$, $C=97°\ 26'\ 29''\cdot0$, $b=99°\ 40'\ 48''\cdot5$, A being less than 180°.
(A $=65°\ 33'\ 10''\cdot2$, $c=100°\ 49'\ 30''\cdot1$.)

(2) $a=95°\ 38'\ 4''\cdot5$, $C=115°\ 36'\ 44''\cdot8$, $b=97°\ 26'\ 29''\cdot0$ ($A<180°$).
(A $=99°\ 40'\ 48''\cdot5$, $c=114°\ 26'\ 49''\cdot8$.)

(3) $a=17°\ 10'\ 47''\cdot3$, $C=150°\ 25'\ 14''\cdot6$, $b=165°\ 4'\ 22''\cdot7$ ($A>180°$).
(A $=270°\ 30'\ 7''\cdot9$, $c=188°\ 22'\ 19''\cdot1$.)

(4) $a=55°\ 42'\ 57''\cdot9$, $c=249°\ 2'\ 7''\cdot0$, $b=144°\ 28'\ 21''\cdot3$ ($A<90°$).
(A $=55°\ 42'\ 57''\cdot9$, $c=223°\ 8'\ 17''\cdot2$.)

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* α Lyræ 1900.

CHAPTER VI

GRAPHICAL METHODS OF SOLUTION

§ 48. **Introductory.**—In plane trigonometry a triangle can be solved (at least roughly) by drawing its sides and angles to scale and simply measuring off the required quantities. In spherical trigonometry it is impossible to do this directly on a plane diagram; various graphical methods of solution have, however, been invented and used. They are inferior, as regards accuracy, to the numerical methods described above; but they are sufficiently accurate for many purposes, and some of them are highly ingenious and interesting. We may roughly divide these methods into two main classes, *Constructive* and *Nomographical*. In the former the solution is obtained by means of a geometrical construction for each separate case; in the latter a more or less elaborate system of graduated lines and families of curves is drawn once for all, to serve for all possible triangles, and the solution is obtained by inspection of this diagram (with, in some cases, the relative motion of parts of a mechanism). We shall consider first the constructive methods.

§ 49. **Monge's Method.***—Imagine the trihedral angle $OABC$ of § 1 constructed of cardboard and situated with the face OBC in the horizontal plane. Through some point P on the upper edge OA draw a plane through the figure to cut OC perpendicularly in F; and through the same point P draw another plane to cut OB perpendicularly in F'. The two cutting planes will intersect each other in a vertical line through P meeting the horizontal plane in some point D, and we can close up the intersected end of the trihedral angle by means of two right-angled triangles PDF and PDF'. If now this closed surface be unfolded by rotating the enclosing triangular faces which are not horizontal about their bases into the horizontal plane, we

* Gaspard Monge (1746–1818) was Professor at the École polytechnique in Paris.

shall have four lines OA'', OB, OC, OA' radiating from O (fig. 12) and containing angles equal to the angles between the lines OA, OB, OC in the erect position. The point P on the plane POC will occupy a position D', such that DFD' is a straight line since it must always lie in the plane cutting OC at right angles. Similarly we have the straight line $DF'D''$ for the face POB. The right-angled triangle PDF will occupy a position GDF such that $G\hat{D}F$ is a right angle and $GF = D'F$.

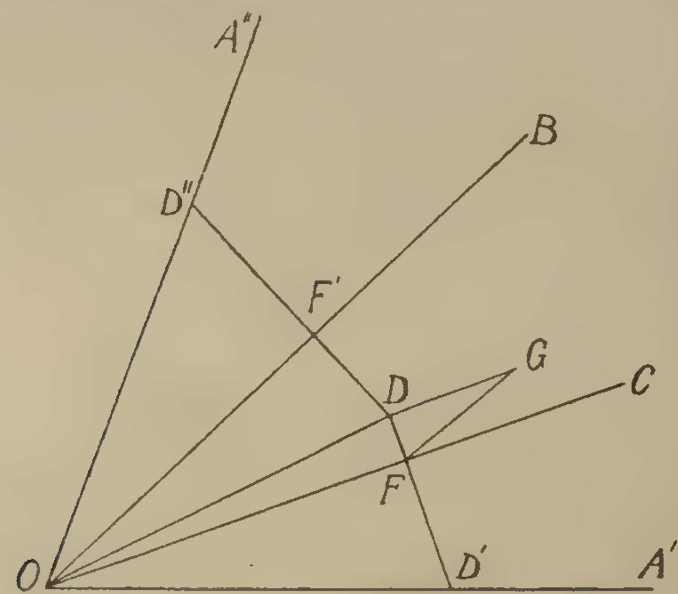

FIG. 12.

We are thus led to the following constructions for solving a triangle graphically according to the parts given:—

(*a*) *Given* a, b, c, *to find* A *or* C.—From a point O (fig. 12) draw four lines OA'', OB, OC, OA' containing the angles $a = A''OB$, $b = BOC$, $c = COA'$. Cut off $OD'' = OD'$ from OA'' and OA', and draw through D'' and D' perpendiculars to OB and OC intersecting in D. Draw DG perpendicular to $D'D$ and make FG equal to FD'. Then $D\hat{F}G$ gives the angle A.

This can be easily verified analytically; for, writing $r = OD$, $r' = OD'' = OD'$, $\theta =$ angle DOF, we have

$$r \cos \theta = OF = r' \cos c.$$

Similarly

$$r \cos (b - \theta) = r' \cos a,$$

or, expanding,

$$\begin{aligned} r \sin b \sin \theta &= r' \cos a - r \cos b \cos \theta \\ &= r' \cos a - r' \cos b \cos c, \end{aligned}$$

which, by the given formula,

$$= r' \sin b \sin c \cos A;$$

$$\therefore \quad r \sin\theta = r' \sin c \cos A,$$

$$i.e. \quad DF = D'F \cos A = FG \cos A$$

$$\therefore \quad \text{angle } DFG = \text{the angle A.}$$

Similarly, by drawing a triangle on DF' with hypotenuse $= D''F'$, we can construct the angle C.

(*b*) *Given b, c, and A, to find a.*—Use the same construction in different order. Draw OB, OC, OA' as before, and at any point F in OC draw FD' perpendicular to OC. Construct the triangle DFG having angle DFG = angle A, $FG = FD'$ and FDG a right angle. This determines D. Draw $DF'D''$ perpendicular to OB such that $OD'' = OD'$, then $D''OB$ is the required side a. The analytical proof is as before.

(*c*) *Given the three sides, to find only one angle B.*—The above construction enables us to find two of the angles at once,

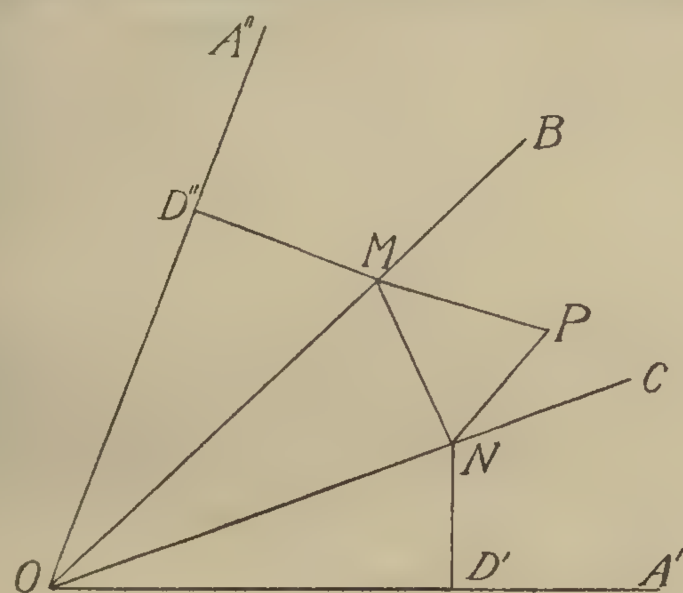

FIG. 13.

A and C. If now with the same lines OA'', OB, OC, OA' laid down, we wish to find the remaining angle A we may use the following construction which is also slightly shorter than the one given above, if, as is often the case, we require to find only one angle.

Through D'' and D' (fig. 13) draw perpendiculars cutting OB, OC, in M and N respectively. Construct the triangle MPN such that $MP = MD''$ and $NP = ND'$; then the angle at P is the required angle B.

For

$$\cos P = (MP^2 + NP^2 - MN^2) \div 2MP \,.\, PN$$
$$= \{MD''^2 + ND'^2 - (OM^2 + ON^2 - 2OM \,.\, ON \cos b)\} \div 2MD'' \,.\, ND'$$
$$= (2OM \,.\, ON \cos b - OD''^2 - OD'^2) \div 2OM \sin a \,.\, ON \sin c$$
$$= \cos b / (\sin a \sin c) - OD''^2 / OM \,.\, ON \,.\, \sin a \sin c$$
$$\therefore \cos b = \frac{OD'' \,.\, OD'}{OM \,.\, ON} + \sin a \sin c \cos P$$
$$= \cos a \cos c + \sin a \sin c \cos P,$$

showing that $\hat{P}$ is equal to the angle B.

The construction is obviously derived from the consideration that the figure $OD''MPND'O$ can be folded about the lines OM, MN, NO so as to form the trihedral angle already discussed.

It will be noticed that the above constructions are for the case (3.1) only.

Example 1.—Given $a=56°\ 40'$, $b=83°\ 13'$, $c=114°\ 30'$, find the angles by Monge's construction. (*Ans.:* A = 48° 31′, B = 62° 56′, C = 125° 19′.)

Example 2.—Given $a=83°\ 12'$, $b=56°\ 40'$, C = 125° 20′, find A and B by graphical construction. (*Ans.:* A = 62° 54′, B = 48° 30′.)

§ 50. **The Emerson-Langley Construction.**—We shall next describe a second method for solving graphically the equation

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

which is due to W. Emerson (1701–1782) and E. M. Langley.*

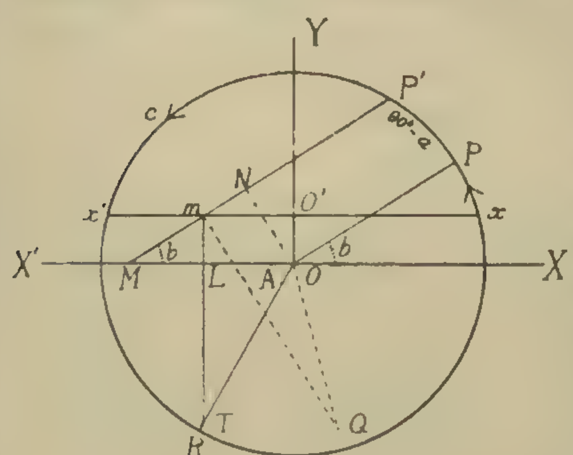

Fig. 14.

Draw a circle $XPX'R$ of unit radius (fig. 14) about the origin. Graduate its circumference to read degrees, minutes, etc., and let the angles XOP, YOx' and $X'OR$ be equal to b, c, and A respectively. Through x' draw $x'O'x$ parallel to the x-axis and cut off $OT = O'x$. Draw TLm parallel to the y-axis meeting $x'x$ in m. Through m draw mP' parallel to OP. Then $P'\hat{O}P$ read off on the circumference is the complement of the side a.

* *Cf.* Langley, *The Graphic Treatment of some Astronomical Problems*, London (1908). It may here be mentioned that Mr Langley has devised many interesting cardboard models representing the solution of spherical triangles.

We have

$$OT = O'x = \sin c$$
$$OL = OT \cos A = \sin c \cos A.$$

Also

$$OL = OM - ML = ON \operatorname{cosec} b - \cos c \cot b$$
$$\therefore\ ON = \cos c \cos b + \sin c \cos A \sin b$$
$$= \cos a \quad \text{by formula given.}$$

But

$$ON = \sin P'\hat{O}P \quad \therefore\ a = 90^\circ - P'\hat{O}P$$

or

$$P'P = 90^\circ - a, \quad \text{as stated.}$$

This construction, therefore, gives us the remaining side when two sides and the contained angle are given.

It is obvious that the same construction gives us the angle A when the three sides are given, for the values of a, b, c determine the points P, P', x', and therefore the point m and the length $OT = O'x$ are given so that the angle A is readily constructed.

Again, if one of the containing sides (b) be required, the values of c and A determine in succession the line $x'x$, the point T, and the point m. If now a parallel linkage * having one arm pivoted at O be adjusted until the other passes through m, and the intercept PP' on the circumference equals $90° - a$, then b is given by the arc PX.

We can construct the angle C at the same time as follows:—Through m draw mQ perpendicular to mP' so that $OQ = NP' = \sin a$. Then

$$P\hat{O}Q = \text{angle C}.$$

For $\cos P\hat{O}Q = \cos(90° + Q) = -\sin Q = -\dfrac{mN}{OQ} = \dfrac{-mN}{\sin a}$. Now mN is the projection of mL and LO where $mL = \cos c$, and LO, by above, $= \cos a \operatorname{cosec} b - \cos c \cot b$.

$$\therefore\ mN = -mL \sin b + LO \cos b$$
$$= -\cos c \sin b + \cos a \cot b - \cos c \cot b \cos b$$
$$= \cos a \cot b - \cos c/\sin b$$

$$\therefore\ \cos P\hat{O}Q = \frac{-mN}{\sin a} = \frac{\cos a \cos b - \cos c}{\sin a \sin b} = \cos C.$$

* The writer is not aware that this additional mechanism was contemplated by the authors.

Example 1.—*Prove the following constructive solution for a spherical triangle whose three sides are given* :—

Let the sides a, b, c, be given, and let C be the part required. Describe a circle, and take any point Y on it. Mark off on the circumference from Y, on the same side of Y, arcs YS, YT, respectively equal to the sides a and b, and on the other side of Y mark off an arc YM equal to a. From T mark off on each side arcs TN, TQ, each equal to c. Draw the chords SM and QN, meeting in P. Bisect SM at D, and with D as centre and DM as radius draw a circle MRS. Draw a perpendicular PR to MPS at P, meeting this circle in R. Draw the radius DR. Then the angle SDR will be equal to the required angle C.

[Imagine the semicircle SRM rotated about the line SM as a hinge, until its plane is perpendicular to the plane of the paper. Let R′ be the new position of R. Then if we consider the sphere of which the circle MYSTQN is a great circle, the points R′, Y, T, are on the surface of this sphere, and the arcs R′Y, YT, TR′ are respectively equal to the given sides a, b, c so that R′YT is equal to the spherical triangle in question. The construction is then deduced without difficulty.]

Example 2.—*Prove that with the construction of Example* 1 *the angle* A *can be obtained in the following way* :—

With centre R and radius equal to $\frac{1}{2}$NQ describe a circle meeting SM in E. Join RE. Then the angle DER is equal to the required angle A.

Example 3.—Prove that in the construction of Example 1, $PD = \sin a \cos C$.

Example 4.—If $a = 62° \, 40'$, $b = 119° \, 0'$, $c = 79° \, 10'$, find the angles by any of the above graphical constructions. (*Ans.*: $A = 50° \, 10'$, $B = 130° \, 50'$, $C = 58° \, 0'$.)

§ 51. **Nomograms : Definition.**—We now proceed to describe the nomographic methods. As stated above (§ 48), a nomogram, in contradistinction to the constructive methods just described, consists essentially of a table of values of a function represented on one or more comparatively elaborate diagrams. When once drawn they give in general more accurate solutions than can be obtained by direct construction. A simple example in which only one diagram is required is the daily weather chart, which enables us to read off directly the temperature and atmospheric pressure at any locality.

The following three methods (§§ 52–55) involve two diagrams, one of them being transparent and sliding over the other, and are derived from consideration of the projection of a triangle drawn on a sphere.

§ 52. **Spherical Networks.**—Let us describe on a sphere the network of parallels of latitude and great circles of longitude of § 5, fig. 1, having N as one of its poles, and imagine a transparent spherical shell closely fitting over this and covered with an exactly similar network, Z being one of its poles. It is plain that by making the arc ZN equal to one side of our triangle,

we can trace the other sides of the triangle, one along each set of meridians, their point of intersection being the third vertex S. The lengths of the sides are read off on the parallels of latitude in each case, and the angles at N and Z are read off where the sides produced cut the corresponding equators. The methods which we shall develop in §§ 53–55 are essentially representations of the above spheres on a plane or planes.

Spherical Trigonometry has its origin in the problem of finding the length of time a heavenly body with a given declination remains above the horizon at a given latitude. This is evidently (§ 44) the problem of solving the triangle ZNS (fig. 9), where S is the position of the body and where ZS is a quadrant. The ancients, having none of the modern trigonometrical formulæ, could only solve the problem graphically by a projection, and for this purpose designed the famous *Analemma* of Ptolemy, which we now proceed to describe.

§ **53. The Analemma.***—Project orthogonally the network having N as pole, on the plane containing N, Z, and the centre of the sphere (the meridian plane). Then the meridians project (fig. 15) into ellipses having NN′ as major axis; and the parallels of latitude, which give the declination, project into straight lines parallel to X′X (since their planes are perpendicular to NN′). In the figure the complication of a large number of lines on a small diagram has been avoided by giving only one of the straight lines ($x'O'x$) and only six of the ellipses which mark off angles of 30° (two hours) at N. With Z the zenith as pole we should have an exactly similar projected network, but as NZ is a variable length, it cannot be drawn fixed upon the other network. The equator of Z, P′OP is represented by a revolving "pointer" turning about O. At sunrise ZS is a quadrant, so that the projection of the sun S is always on the pointer if $N\hat{O}P$ be equal to the latitude of the place. Further, if we read off $X'x'$ or Xx equal to the declination (south in the figure), we have the parallel $x'x$ on which the sun lies, so that S is now determined; and by means of the ellipse (hour-angle) which passes through S we read off on X′X the time before noon at which sunrise occurs. In the figure this is indicated as

* For a historical account see Delambre's *Histoire de l'astronomie ancienne*, tome ii. pp. 458–503.

five hours, *i.e.* 7 a.m., so that the length of day is ten hours. The ellipse passing through S and Z gives the angle NZS, *i.e.* the *azimuth* (measured from the north) at which the sun rises. If the analemma is to be used at a fixed latitude and for this problem only, the ellipses of which ZS is one, might be permanently drawn on the figure; but since their equator P′OP is given, we can equally well mark their intersections upon it and read off the azimuth directly. This graduation on the pointer

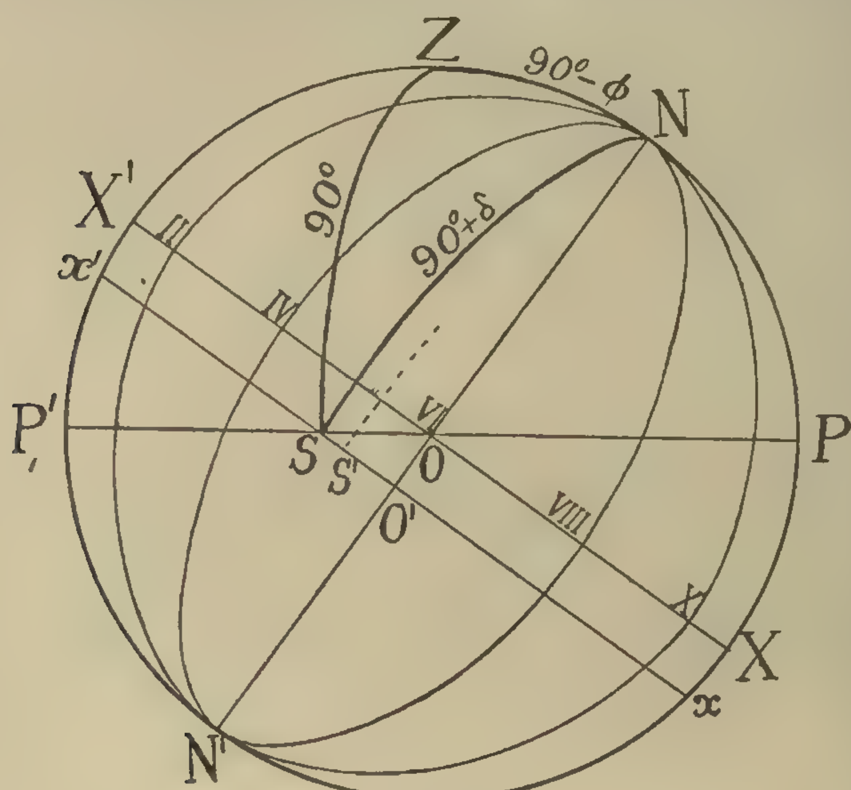

FIG. 15.

will plainly be $r \sin A$, where A is the azimuth measured from its east-west direction—the meridian marked VI. On the early analemmas this was effected by drawing on the figure a family of circles about O as centre, their radii $r \sin A$ being obtained graphically from a right-angled plane triangle of angle A.

We can now extend the construction so as to solve triangles of any sides. Let two of the sides be $90° - \phi$, $90° + \delta$, as before, but let the third side be, say, $90° + \alpha$. Take off with the dividers from OP a length $= r \sin \alpha$, and so find a point S′ between S and x such that the perpendicular from it to P′P equals this length. Then the ellipse through S′ gives the new angle at N.

For $O'S = r' \cos Z\hat{N}S$ (where r' is the unprojected distance from S to the axis $NN') = r \cos \delta \cos Z\hat{N}S$.

Similarly

$$O'S' = r \cos \delta \cos Z\hat{N}S'.$$

$$\therefore \qquad SS' = r \cos \delta \,(\cos Z\hat{N}S - \cos Z\hat{N}S')$$

$$= r \cos \delta \left\{ \frac{\cos ZS - \cos ZN \cos NS}{\sin ZN \sin NS} - \frac{\cos ZS' - \cos ZN \cos NS'}{\sin ZN \sin NS'} \right\}$$

$$= \frac{r \cos \delta \cos (90^\circ + a)}{\sin (90^\circ - \phi) \sin (90^\circ + \delta)} = \frac{r \sin a}{\cos \phi} = \frac{r \sin a}{\sin O\hat{S}S'},$$

which agrees with the construction.

§ 54. **Chauvenet's Solver.**—We owe to Chauvenet a device whereby all the parts of a triangle except one angle are read off at one setting of a simple mechanism. From a point on the two spherical networks described in § 52 which is 90° from both N and Z, project (stereographically) the two spheres on to two superimposed meridian planes. Let the one nearer to the point of projection be transparent, so that both projected systems are visible. The projection will cover both planes to infinity, but the remoter hemispheres will project into circular areas having the projection of the centre as common centre, and having the projections of N and Z on the circumferences. By this projection all circles, great or small, project into circles, and the meridian circles all pass through the projections of their respective poles, the latitude circles cutting these orthogonally and having their centres on the corresponding axis. The continuous lines of fig. 16 show the projection of the N system, while the broken lines represent the Z or transparent system. The axes are inclined in the figure at about 38°. If R be the radius of the figure, we can easily show that the radius r of the projection of a small circle of latitude ϕ is given by $r = \mathrm{R} \cot \phi$, and that the centre of the projected circle is at a distance $\mathrm{R}/\sin \phi$ from O along the axis NS. Also if θ be the angle made by one of the meridian circles with the plane of projection, then the projected circle has a radius $= \mathrm{R} \sec \theta$, and has its centre at a point distant $\mathrm{R} \tan \theta$ from O and passes through N and S. From these data the circles are easily constructed.

If now we rotate the transparent network having Z as pole

about the common centre, until ZN measures one side of the triangle, then the other sides and angles can be traced along one or other of the networks and the unknown parts read off on scales along the equators and round the circumferences.

In fig. 16 a triangle $ZN\sigma$ (σ not shown) is picked out in heavy lines having an angle at $Z=135°$, the side $ZN=38°$, and the side $Z\sigma=45°$. A device of this kind has been placed on the

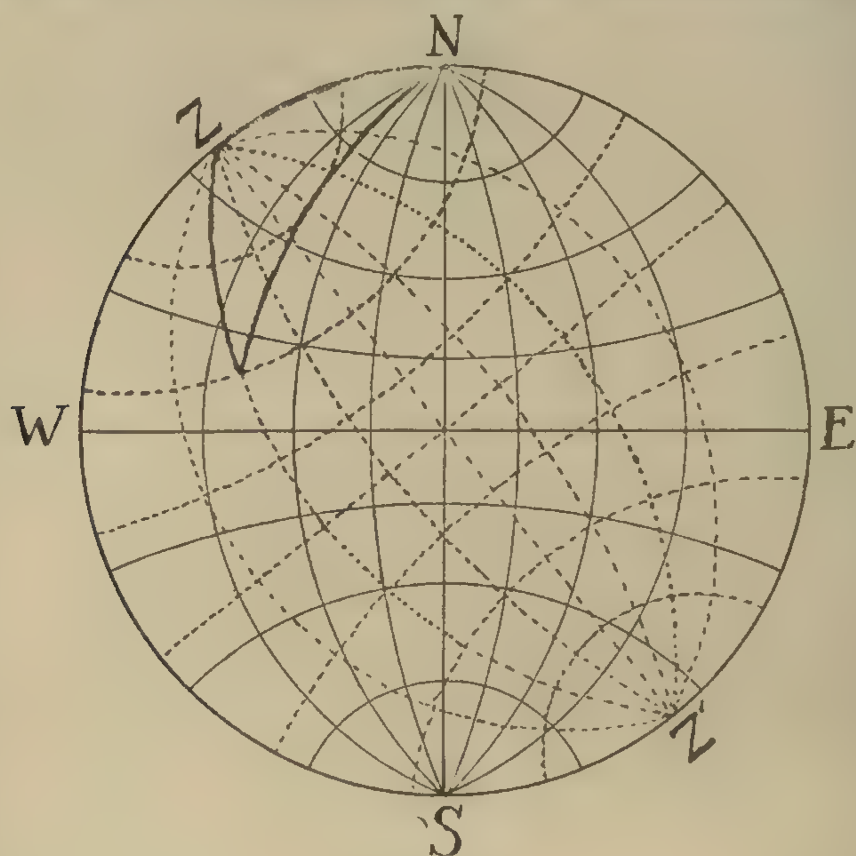

FIG. 16.

market * in which the circles have a diameter of about $8\frac{1}{2}$ inches, and enable one to read to about half a degree; this is sufficiently accurate for certain navigation problems, *e.g.* finding the course for great circle sailing (§ 43). By means of it we find that for the above triangle $N\sigma=75\frac{1}{2}°$ and $ZN\sigma=31°$.

§ **55. Cylindrical Projection.**†—Imagine two closely fitting right circular cylinders, the inner one being transparent, touching the two spheres of § 52 along the great circle ZN. If we project the two networks having Z and N as poles on to

* It is published by Reimer of Berlin, after the designs of E. Kohlschütter. A larger one, 18 inches in diameter, constructed by Mr W. B. Blaikie of Edinburgh, was used by Sir F. W. Dyson in researches on the systematic motions of the stars, *cf. Month. Not., R.A.S.*, vol. lxx. (1910), p. 416.

† The author believes that this method has not been published previously.

these tangential cylinders from the common centre (*gnomonically*) and then remove the spheres, we could evidently solve any triangle in the following manner. Rotate the cylinders until the arc joining ZN is equal to one of the given sides, then trace out the other two given parts along the corresponding networks and read off the other parts (all except the angle opposite the given side) from them. If we cut the cylinders

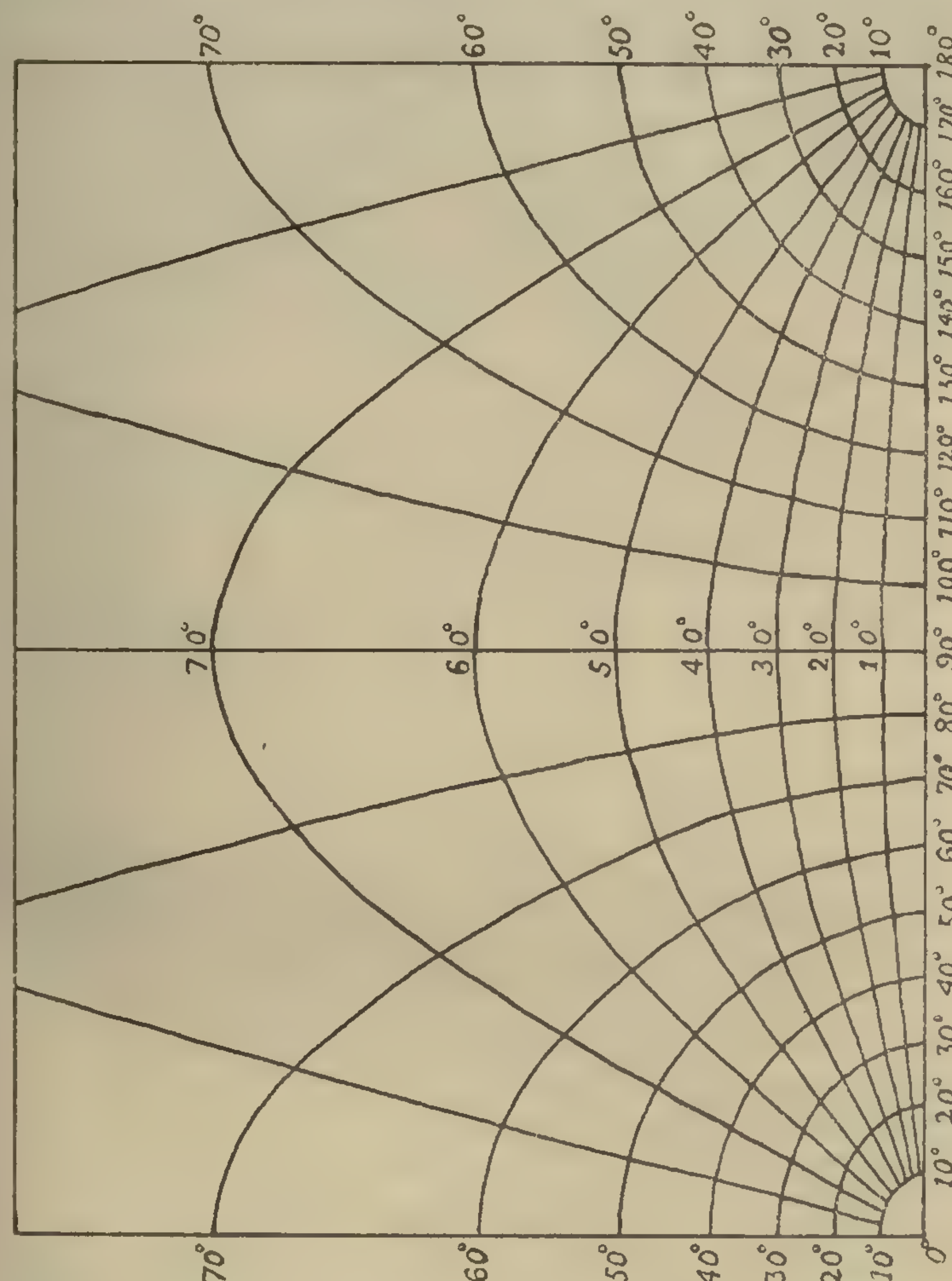

Fig. 17.

open along a generator and flatten them out on a plane, we can vary the side ZN by a simple translation. We can now imagine the two superimposed plane networks enlarged to any extent and retain *any* area of one of them, say, a square foot of the lower one, retaining also that part of the transparent one which is necessary for solving triangles to the required accuracy which come within its scope.

In fig. 17 half the circumference of one cylinder is given, allowing ZN to vary between 0° and 180°. The other (transparent) sheet is supposed to be placed upon it

with the axes coincident but displaced relatively to each other by an amount equal to one side as read off along the base. The equation of the network is

$$\cos^2 x = (1+y^2)\cos^2\phi,$$

where ϕ is the side NS, *i.e.* the polar distance; and

$$y = \sin x \tan\theta$$

where θ is the angle ZNS.

§ 56. **Slide Rule for Sine Formula.**—The formula $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}$ which, being in product form, is convenient for logarithmic calculation can be adapted to graphical calculation as follows:—

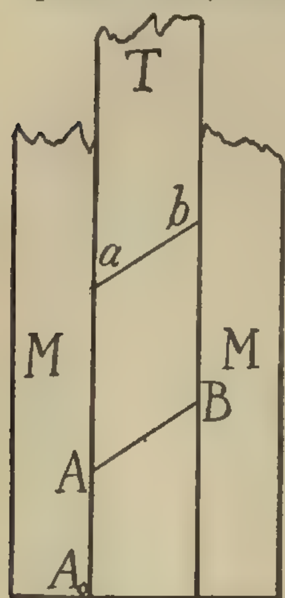

Fig. 18.

Construct a tongue T to slide in a groove (fig. 18) in a framework M. Suppose, for a moment, the tongue made flush with the framework at one end A_0; and let B_0 be *any* point on the other line of contact. On the framework construct the scales $A_0A = \mu \log \sin A$, $B_0B = \mu \log \sin B$ for a succession of values of A and B where μ is *any* arbitrary constant. On the tongue construct the scales $A_0a = \mu \log \sin a + \alpha$, $B_0b = \mu \log \sin b + \alpha$. Then $Aa = A_0A - A_0a = \mu \log \frac{\sin A}{\sin a} - \alpha$. Similarly $Bb = \mu \log \frac{\sin B}{\sin b} - \alpha$, so that, since these are equal, ab and AB are parallel. If therefore the tongue T be moved in the groove until the points reading the given values of B and b coincide, then the required value of a is read off on its scale opposite the given value of A. The constants μ, α and the position of B_0 are at our disposal, so that we can graduate the scales to read to any desired degree of accuracy, and can bring the required parts of the A and a scales as well as those of the A and B scales opposite each other.

§ 57. **Straight-Line Nomograms.**—The following nomograms, due to d'Ocagne, are simpler than the preceding ones in that the sliding (transparent) diagram has in each case been reduced to a *straight line joining three points.** We shall show that a nomogram can be constructed for each of the four-variable equations which solve the cases (2.2), (3.1), and (4.0); these we shall take in turn in the following sections. Each of them has two parallel straight lines A_0u, B_0v (fig. 19) along which are graduated scales. By means of these scales, points are determined on A_0u, B_0v (one on each scale) to represent the values of two of the four variables. A third point (not on either of these lines) is determined to represent by its position the remaining two

* The whole subject of nomography has been developed extensively since 1884 by Maurice d'Ocagne, now Professor at the École polytechnique in Paris. Figs. 20–23 are reproductions of diagrams due to him. *Cf. S. M. F. Bull.* 32 (1904), p. 196.

variables, and it is the fundamental property of the nomogram that the straight line joining the two points on the scales passes through this third point. The details of construction and application in each case are as follows:—

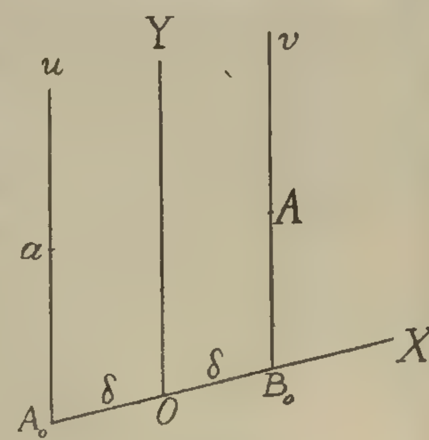

FIG. 19.

§ 58. **The Case (2.2).**—*To determine the value of the fourth variable in the equation*

$$\sin A \sin b = \sin B \sin a$$

when the values of three are known.

Along the opposite sides of the straight line A_0u (fig. 20) construct two scales, viz. $A_0A = \mu_1 \sin A$ and $A_0a = \mu_2 \sin a$ for a succession of values of A and a, where μ_1 and μ_2 are any arbitrary constants. Along opposite sides of the parallel straight line B_0v construct

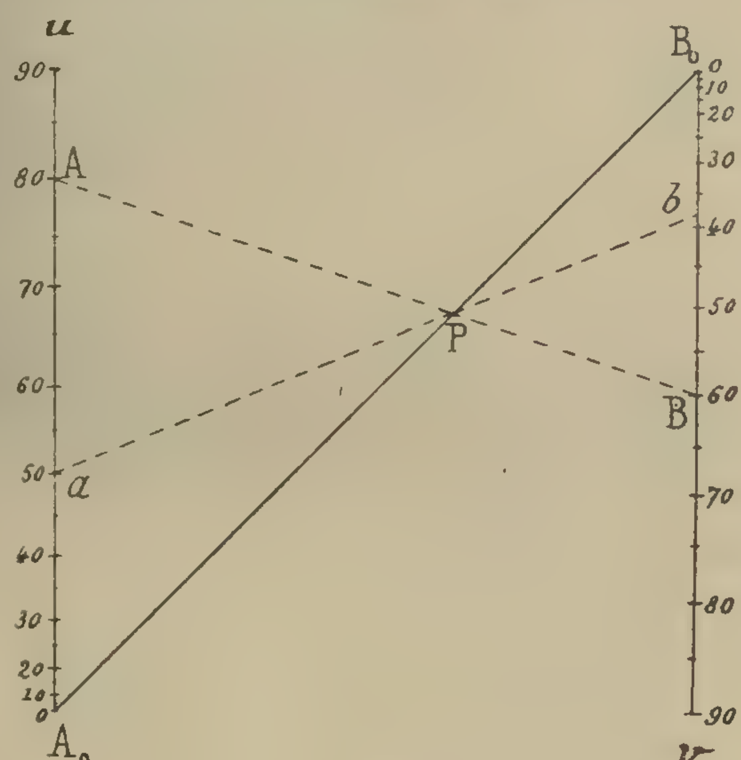

FIG. 20.

the two scales $B_0B = \mu_3 \sin B$ and $B_0b = \mu_4 \sin b$ for a succession of values of B and b, where μ_3 and μ_4 are any arbitrary constants satisfying the equation $\mu_1\mu_4 = \mu_2\mu_3$. Join A_0B_0. Then the method of application is as follows, supposing that a is the required

variable. Join the points A, B, corresponding to the given values of A and B by a straight line cutting A_0B_0 in P. Join P to the given point on the b scale by a line cutting A_0u in a. The scale A_0a gives us the required value of a.

For, by similar triangles,

$$\frac{A_0a}{A_0A}=\frac{B_0b}{B_0B}, \quad i.e. \quad \frac{\mu_2 \sin a}{\mu_1 \sin A}=\frac{\mu_4 \sin b}{\mu_3 \sin B},$$

which is true by the given equation and the equation $\frac{\mu_2}{\mu_1}=\frac{\mu_4}{\mu_3}$.

Three of the constants are at our disposal, and we can choose the positions of A_0 and B_0 arbitrarily. We can thus construct our scales to read to any degree of accuracy and bring the points A, a, B, b more or less opposite to each other, if (as is the case in many practical problems) the variables are limited to a moderate range of values.

§ 59. **The Case (3.1).**—*To determine the value of the fourth variable in*

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad . \quad . \quad . \quad (1)$$

when any three are given.

Along the lines A_0u and B_0v (fig. 19), whose equations are $x=-\delta$ and $x=+\delta$, construct the two scales A_0a and B_0b such that

$$u=A_0a=\mu_1 \cos a$$
$$v=B_0b=-\mu_2 \cos A,$$

where μ_1 and μ_2 are arbitrary constants. We shall prove that, provided a and A continue to satisfy (1), the straight line aA always passes through the point whose coordinates are

$$x=\delta\frac{\mu_1 \sin b \sin c-\mu_2}{\mu_1 \sin b \sin c+\mu_2} \quad . \quad . \quad . \quad . \quad (2)$$

$$y=\frac{\mu_1\mu_2 \cos b \cos c}{\mu_1 \sin b \sin c+\mu_2} \quad . \quad . \quad . \quad . \quad (3)$$

For, substituting for $\cos b \cos c$ from (1) and writing $\cos a=u/\mu_1$, $\cos A=-v/\mu_2$, (3) can be written

$$y=\frac{\mu_2 u+\mu_1 v \sin b \sin c}{\mu_2+\mu_1 \sin b \sin c} \quad . \quad . \quad . \quad . \quad (4)$$

Again, by (2) we have

$$\mu_1 \sin b \sin c=\mu_2\frac{\delta+x}{\delta-x} \quad . \quad . \quad . \quad . \quad (5)$$

so that, substituting in (4), we easily obtain

$$2\delta y = \delta(u+v) - x(u-v) \qquad . \quad . \quad . \quad . \quad (6)$$

Now the condition that the three points $(-\delta, u)$, (x, y), (δ, v) be collinear is

$$\begin{vmatrix} x & y & 1 \\ -\delta & u & 1 \\ \delta & v & 1 \end{vmatrix} = 0 \qquad . \quad . \quad . \quad . \quad (7)$$

This is equivalent to (6), so that the collinearity is proved.

If now we treat (2) and (3) as freedom equations (with two parameters b and c), we can draw two families of curves, one by keeping b constant, the other by keeping c constant. These coincide with each other on account of the symmetry of (2) and (3). To any particular value of b corresponds a certain curve (a conic) and to any value of c another conic; the two intersect in the point (2) (3), which lies on the line joining aA.

To show that these curves are conic sections we may proceed as follows:—

Substituting from (5), we can write (3) in the form

$$\mu_1\mu_2 \cos b \cos c = y\left(\mu_2 \frac{\delta + x}{\delta - x} + \mu_2\right)$$

i.e.

$$\mu_1 \cos c = \frac{2\delta y}{\delta - x} \sec b \qquad . \quad . \quad . \quad . \quad (8)$$

$\therefore$ since

$$\mu_1 \sin c = \mu_2 \frac{\delta + x}{\delta - x} \operatorname{cosec} b$$

we have, on squaring and adding,

$$\mu_1^2(\delta - x)^2 = \mu_2^2(\delta + x)^2 \operatorname{cosec}^2 b + 4\delta^2 y^2 \sec^2 b \qquad . \quad . \quad (9)$$

which represents a family of ellipses having b as parameter, provided $\mu_2 \geqslant \mu_1$. Eliminating b instead of c, we obtain the same family of ellipses with c as parameter.

We can write (9) in the form

$$4\delta^2 y^2 \tan^4 b + \{4\delta^2 y^2 + \mu_2^2(\delta + x)^2 - \mu_1^2(\delta - x)^2\} \tan^2 b + \mu_2^2(\delta + x)^2 = 0 \quad (10)$$

and regarding (10) as a quadratic in $\tan^2 b$ its discriminant gives us the envelope of the family of ellipses (9).

The discriminant reduces to

$$4\delta^2 y^2 = \mu_1^2(\delta - x)^2 \pm 4\mu_2 \delta y(\delta + x) - \mu_2^2(\delta + x)^2 \qquad . \quad . \quad (11)$$

which represents four straight lines forming a quadrilateral; the equations of the sides are

$$\left.\begin{aligned} 2\delta y &= x(\mu_2 - \mu_1) + \delta(\mu_2 + \mu_1) \\ 2\delta y &= x(\mu_2 + \mu_1) + \delta(\mu_2 - \mu_1) \\ 2\delta y &= -x(\mu_2 - \mu_1) - \delta(\mu_2 + \mu_1) \\ 2\delta y &= -x(\mu_2 + \mu_1) - \delta(\mu_2 - \ _1) \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

Each of the ellipses is inscribed in this quadrilateral.

The accompanying figure (fig. 21) corresponds to the case in

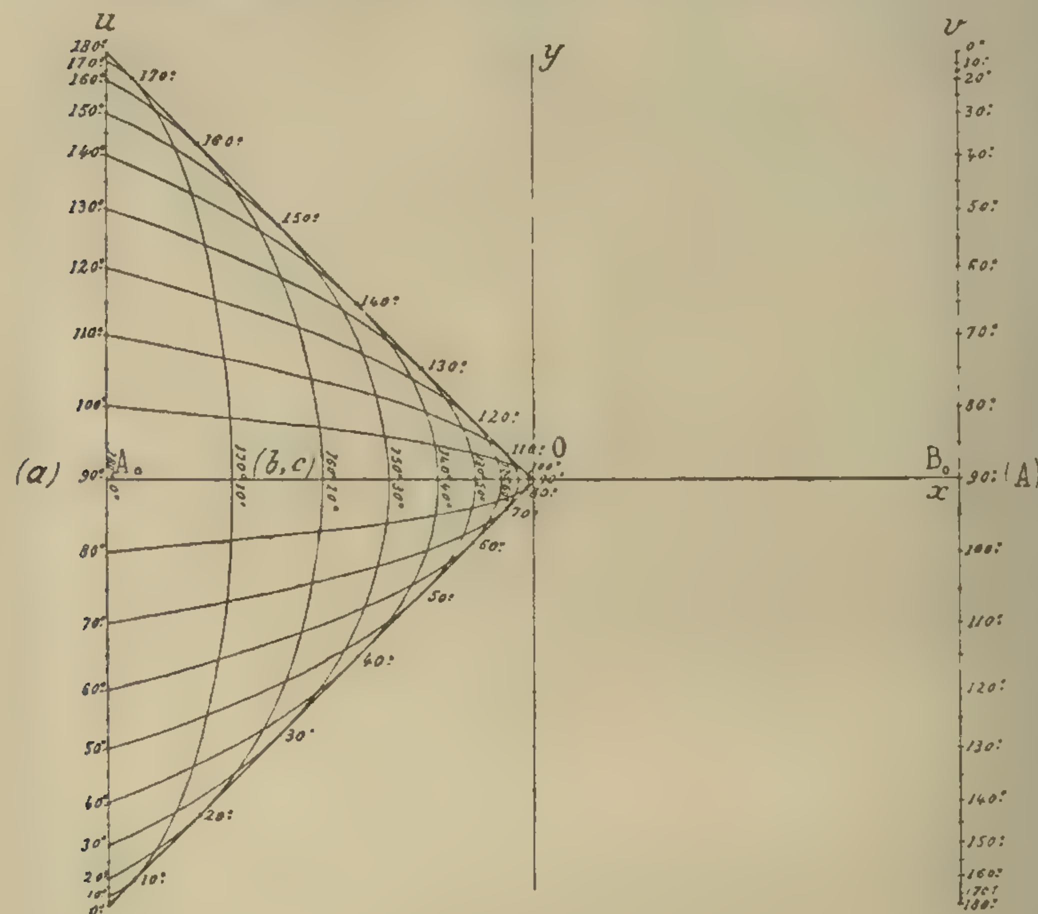

Fig. 21.

which the axes are rectangular and $\mu_1 = \mu_2$. Two of the sides of the enveloping quadrilateral (the second and fourth of group (12)) become $y = \frac{\mu}{\delta}x$, and $y = -\frac{\mu}{\delta}x$. As an example, suppose $A = 60°$, $b = 10°$, and $c = 40°$. We find the two curves on the (b, c) scale reading 10° and 40° and find their two intersections; joining these by a straight edge to the point 60° on the A scale and producing it, we get the two possible values of a, viz. 134° and 36°, other conditions deciding which we need.

§ 60. **Particular Application.**—An interesting and practical special application of this nomogram is quoted by D'Ocagne.* It was required to prepare the observations taken with the equatorial telescope at the Paris Observatory, for which

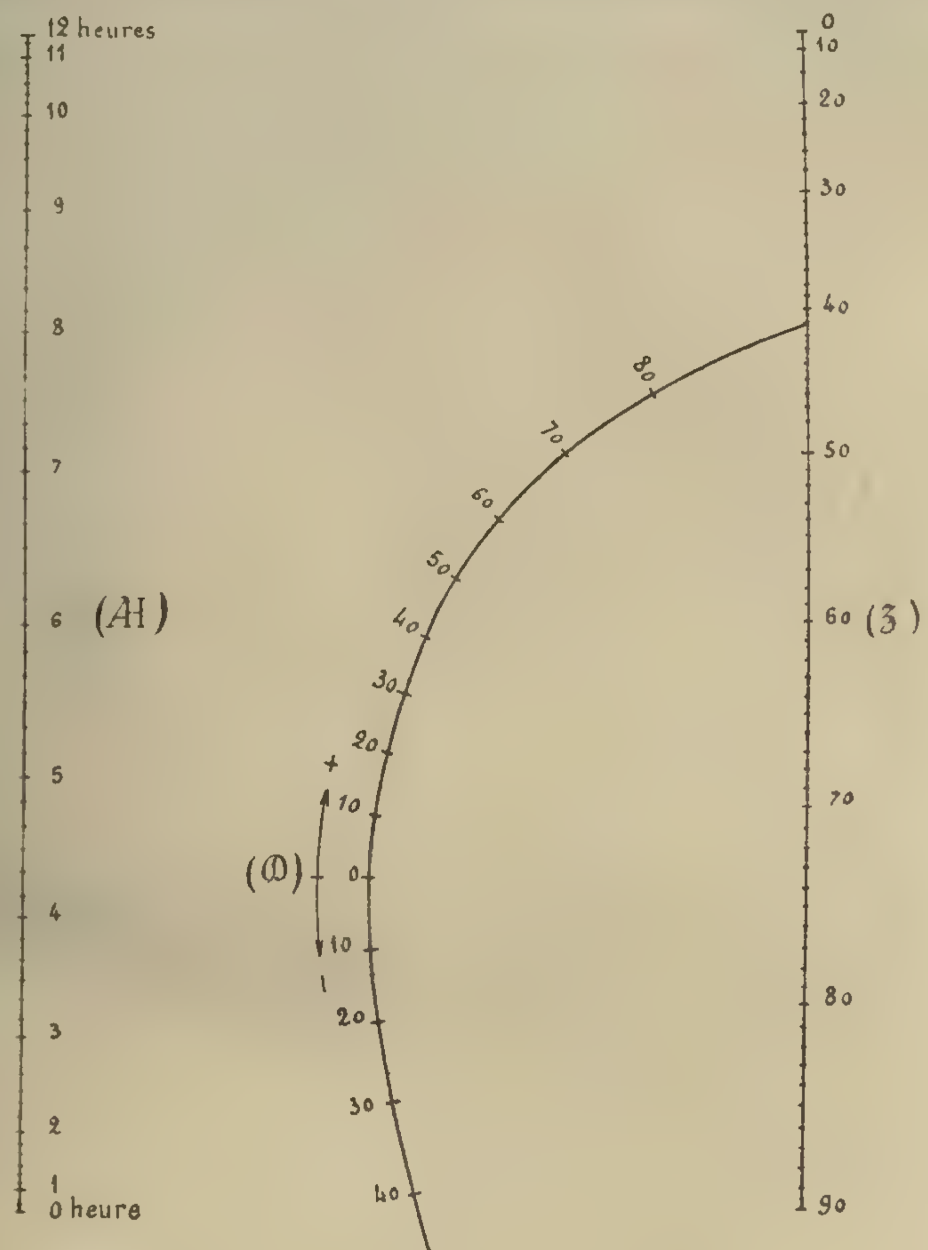

FIG. 22.

$\phi = 48° \ 50' \ 11''$. We have an equation of the form (3.1) connecting the variables zenith distance (Z) (SZ in fig. 9), declination D, and hour-angle Æ (h of fig. 9). The side $ZP = (90° - \phi)$ is constant. Therefore, writing in equation (1) of last section $a = z$, $b = 90° - \phi$, $c = 90° - D$, $A = Æ$, we see that the two scales at

* *Comptes rendus*, t. cxxxv., 1902.

the sides and the ellipse $b=(90°-\phi)$ out of the network are sufficient. This ellipse must be graduated at the points where the various c ellipses if drawn would intersect it. Fig. 22 gives the resultant nomogram where μ_1 has been taken * $=2\mu_2$, and the diagram is sheared so as to bring the required parts of the scales opposite each other. The figure must be inverted to make it agree with fig. 21.

§ **61. The Case (4.0).**—*To determine the values of the fourth variable in*

$$\cos B \cos a = \cot c \sin a - \cot C \sin B \quad . \quad . \quad . \quad (1)$$

connecting the four consecutive parts c, B, a, C, when the values of any *three are given.*

Along the two lines A_0u and B_0v (fig. 19) construct the two scales $A_0a=\mu_1 \cot c=u$; $B_0A=-\mu_2 \cot C=v$.

Then we can show that, provided c and C continue to satisfy equation (1), the straight line Aa always passes through the point given by

$$x=\delta\frac{\mu_1 \sin B-\mu_2 \sin a}{\mu_1 \sin B+\mu_2 \sin a} \quad . \quad . \quad . \quad . \quad (2)$$

$$y=\frac{\mu_1\mu_2 \cos B \cos a}{\mu_1 \sin B+\mu_2 \sin a} \quad . \quad . \quad . \quad . \quad (3)$$

For, substituting for cos B cos a from (1) and writing $\cot c=u/\mu_1$, $\cot C=-v/\mu_2$, (3) can be written

$$y=\frac{\mu_1 v \sin B+\mu_2 u \sin a}{\mu_1 \sin B+\mu_2 \sin a} \quad . \quad . \quad . \quad . \quad (4)$$

Again, by (2) we have

$$\mu_1 \sin B=\frac{\delta+x}{\delta-x}\mu_2 \sin a \quad . \quad . \quad . \quad . \quad (5)$$

so that, substituting in (4), we easily obtain

$$2\delta y=\delta(u+v)-x(u-v) \quad . \quad . \quad . \quad . \quad (6)$$

which, as in § 59, shows that the three points $(-\delta, u)$, (x, y), and (δ, v) are collinear.

If now with these two equations we construct as, in the preceding case (§ 59), the two families of curves, one keeping a

* This changes the ellipse into a hyperbola, *cf.* equation (9) of last section.

constant, the other keeping B constant, we obtain a doubly infinite set of conic sections such that the two curves corresponding to the given values of a and B intersect in the point (2) (3) which lies on the line joining aA (fig. 19).

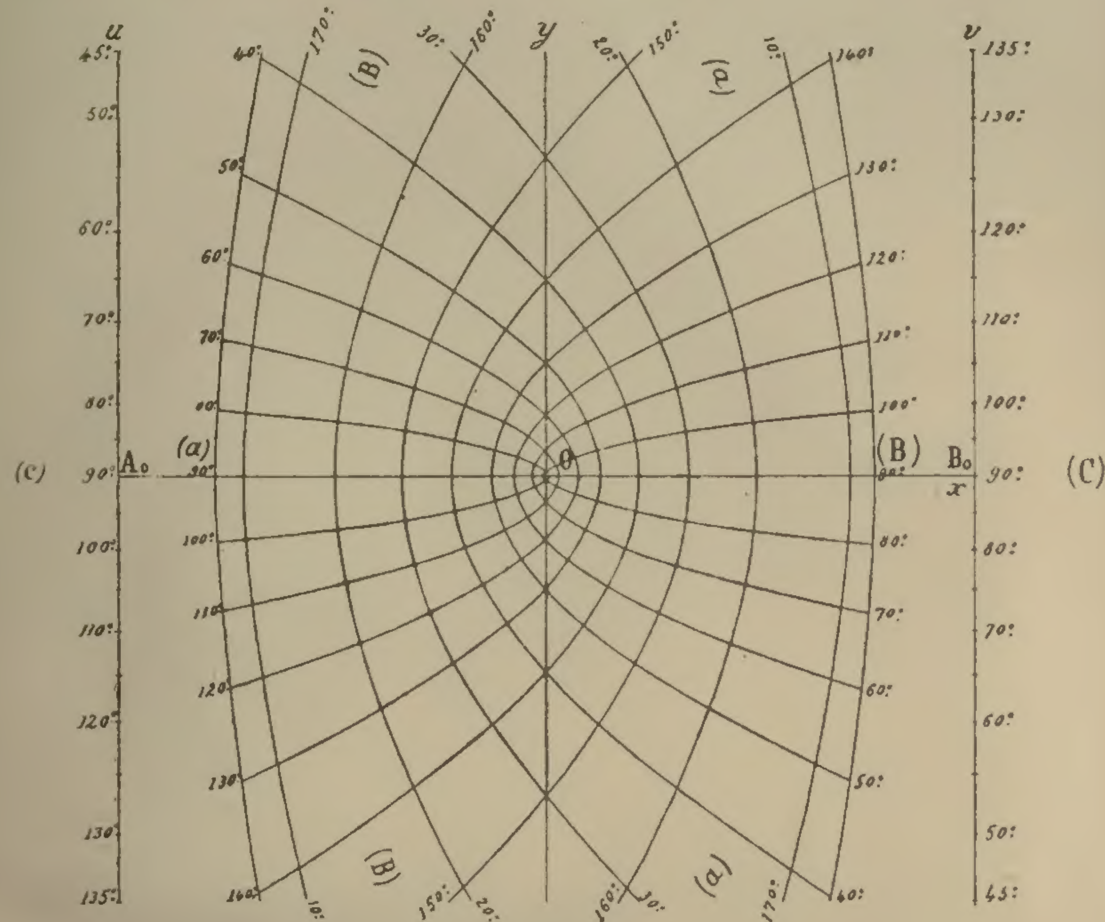

FIG. 23.

To show that these curves are conics, we may proceed as follows: Substituting from (5), we can write (3) in the form

$$\mu_1\mu_2 \cos B \cos a = y\left(\frac{\delta + x}{\delta - x}\mu_2 \sin a + \mu_2 \sin a\right)$$

i.e.
$$\mu_1 \cos B = \frac{2\delta y}{\delta - x} \tan a$$

$\therefore$ since
$$\mu_1 \sin B = \frac{\delta + x}{\delta - x}\mu_2 \sin a$$

we have, on squaring and adding,

$$\mu_1{}^2(\delta - x)^2 = (\delta + x)^2\mu_2{}^2 \sin^2 a + 4\delta^2 y^2 \tan^2 a,$$

a family of hyperbolæ of parameter a, provided $\mu_1 \geqslant \mu_2$.

Eliminating a instead of B, we have the other family,

$$\mu_1{}^2(\delta - x)^2 = (\delta + x)^2\mu_2{}^2 \operatorname{cosec}^2 B - 4\delta^2 y^2 \sec^2 B$$

with parameter B. They are hyperbolæ provided $\mu_2 \geqslant \mu_1$.

The accompanying figure (p. 65) has been drawn for the case in which the axes are rectangular and $\mu_1 = \mu_2$, so that both sets of curves are hyperbolæ. The method of using is the same as in the preceding case. Quite a small nomogram of this kind would solve with sufficient accuracy the problem of great-circle sailing (§ 43).

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